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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ... LAGRANGE

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On Certain Steady Flows of General Fluids

BERNARD D. COLEMAN & WALTER NOLL

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Introduction

In a recent article on the mechanical behavior of continuous media [*1*], the physical concept of a *fluid* was given a mathematical definition. We believe that this definition covers almost all real fluids (whenever thermal and other non-mechanical effects can be disregarded) and is more general than any proposed previously because it accounts for all hereditary effects including stress relaxation. The general fluids covered by this definition may even exhibit, in some situations, physical phenomena which are usually attributed to solids.

Here we show that certain steady flow problems can be solved assuming only the definition of a general fluid and incompressibility. The class of problems considered includes most of the classical flows: simple shearing flow, channel flow (*i.e.* flow between fixed plates), Poiseuille flow, and Couette flow. It also includes torsional and other flows for which we do not work out the details, because, under reasonable body forces, such flows are not compatible with the dynamical equations unless inertia is neglected.

The solutions obtained here are in terms of three unspecified real functions of one variable which depend on the particular material; we call them *material functions*. Our solutions are simply related. If experimental measurements are combined with any one of the solutions to determine the three material functions, then complete stress and velocity profiles can be predicted for the other flows. Thus, our work really establishes correspondence principles for the flows considered. It is hoped that our results may help experimental rheologists to rationalize the treatment of their data by giving precise meaning to such concepts as "shear-dependent viscosity".

If one uses a special theory, either molecular or phenomenological, to solve a particular flow problem of our class (say, simple shearing flow), then our three material functions can be readily identified*. The solutions to the other flow problems can then be obtained from the formulae given here without further reference to the original special theory**.

It should be pointed out that our three material functions do not determine the behavior of the fluid in all experimental situations, since the functional that predicts the response of a material to all situations may contain vastly more information than the three material functions introduced here.

Some of the solutions and relations obtained here have appeared in the literature, but the derivations given have rested upon special assumptions about the nature of the fluid considered. Of these previous investigations, the most general is Rivlin's study of flows in fluids of the differential type [*4*]. Such fluids do not have long range memory and hence constitute only a special class of the general fluids considered here. It turns out, however, that Rivlin's solutions (including those for helical flow, which will be discussed in a future note) remain valid for general fluids.

This paper can be read independently of reference [*1*] if the reader is willing to accept equations (1.1)—(1.3) below.

Notation. We denote vectors and points by bold face Latin or Greek minuscules: $\boldsymbol{v}$, $\boldsymbol{x}$, $\boldsymbol{\xi}$.

Second order tensors are always denoted by light face Latin majuscules: A, B, I, Q. The transpose of a tensor Q is written Q^T. The unit or "metric" tensor is denoted by I. The symbol t_{ij} denotes the *physical* components of T relative to an orthogonal, but not necessarily Cartesian, coordinate system, except when a statement is made to the contrary. We use the summation convention throughout. The matrix of the physical components of T is denoted by (T) or $\|t_{ij}\|$, and the trace of T is written $\operatorname{tr} T$.

Partial differentiation is indicated as follows: $\partial_y = \partial/\partial y$ or $\partial_i = \partial/\partial x_i$.

1. Basic concepts

In a general fluid ([*1*], §§ 21 and 22) the stress tensor S at time t depends on the history of the motion up to t and is given by a constitutive equation which may be stated in the form (22.12) of [*1*]. In this paper we discuss only isochoric motions; we do not make the dependence of S on the density $\varrho(t)$ explicit, and we write

$$S(t) = \mathfrak{H}(C_t^t). \tag{1.1}$$

Here $\mathfrak{H}$ is a functional of the tensor-valued function C_t^t and satisfies the relation

$$Q\,\mathfrak{H}(C_t^t)\,Q^T = \mathfrak{H}(Q\,C_t^t\,Q^T) \tag{1.2}$$

for any constant orthogonal tensor Q (*cf.* [*1*], (21.7)). The function C_t^t is the history of the right Cauchy-Green tensor taken with respect to the configuration at time t and is defined by

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* A summary of results pertaining to the normal stress effect for special theories has been presented by Markovitz [*2*]. The present investigation gives a proof of his equations (12)—(16), which he offered as conjectures.

** For example, much of the work in Chapter IV of [*3*] could have been avoided if the present results had been known to the author at the time.

$$C_t^t(s) = C_t(t+s) = F_t^T(t+s)\, F_t(t+s), \qquad s \leqq 0, \tag{1.3}$$

where $F_t(t+s)$ is the displacement gradient at time $t+s$ relative to the configuration at time t (*cf.* [*1*], §§ 6 and 7).

In this paper we consider homogeneous *incompressible* fluids. The stress is then determined by the history of the motion only up to a hydrostatic pressure p, and (1.1) must be replaced by

$$T(t) = \mathfrak{H}(C_t^t), \tag{1.4}$$

where

$$T = S + pI \tag{1.5}$$

is called the *extra-stress*. Moreover, the tensor-valued functional $\mathfrak{H}$ is determined only up to an arbitrary scalar-valued functional of C_t^t. We remove this indeterminacy by the normalization

$$\operatorname{tr} T = \operatorname{tr}[\mathfrak{H}(C_t^t)] = 0. \tag{1.6}$$

The pressure p then reduces to the mean pressure

$$p = -\tfrac{1}{3} \operatorname{tr} S, \tag{1.7}$$

and T is the deviatoric part of S. We note that the extra stress T must vanish if the motion reduces to a state of rest, in which case we have $C_t^t(s) = I = \text{const}$. Hence

$$0 = \mathfrak{H}(I). \tag{1.8}$$

Consider a particle which at time t finds itself at the position $\boldsymbol{x}$, with general coordinates x^i, and which was at time $t+s$, $s \leqq 0$, at the point $\boldsymbol{\xi}$, with coordinates ξ^i. For the dependence of $\boldsymbol{\xi}$ on $\boldsymbol{x}$, t, and $t+s$, we write

$$\boldsymbol{\xi} = \boldsymbol{\vartheta}_t(\boldsymbol{x}, t+s). \tag{1.9}$$

The function $\boldsymbol{\vartheta}_t$, with coordinates $\vartheta^i_{(t)}$, is the displacement function relative to the configuration at time t.

It follows from (1.3) that the covariant components of $C_t^t(s)$, relative to the natural basis at $\boldsymbol{x}$ of the coordinate system, are

$$c^{(t)}_{(t)\,ij}(s) = f^k_{(t)\,i}(t+s)\, f^l_{(t)\,j}(t+s)\, g_{kl}(t+s), \tag{1.10}$$

where

$$f^i_{(t)\,j}(t+s) = \partial_j [\vartheta^i_{(t)}(\boldsymbol{x}, t+s)], \tag{1.11}$$

and $g_{kl}(t+s)$ denotes the covariant components of the unit tensor I at $\boldsymbol{\xi} = \boldsymbol{\vartheta}_t(\boldsymbol{x}, t+s)$. If the velocity $\boldsymbol{v}(\boldsymbol{\xi}, t+s)$ is given as a function of the position $\boldsymbol{\xi}$ and the time $t+s$, the displacement function

$$\boldsymbol{\xi}(s) = \boldsymbol{\vartheta}_t(\boldsymbol{x}, t+s) \tag{1.12}$$

is that solution of the differential equation

$$\frac{d\boldsymbol{\xi}}{ds} = \boldsymbol{v}(\boldsymbol{\xi}, t+s) \tag{1.13}$$

which satisfies the initial condition

$$\boldsymbol{\xi}(0) = \boldsymbol{\vartheta}_t(\boldsymbol{x}, t) = \boldsymbol{x}. \tag{1.14}$$

If the motion is a steady flow, the velocity $\boldsymbol{v}(\boldsymbol{\xi})$ depends on only the position $\boldsymbol{\xi}$ and not on the time $t+s$. In this case $\boldsymbol{\vartheta}_t(\boldsymbol{x}, t+s)$, and hence also $F_t(t+s)$ and $C_t^t(s)$, are independent of the reference time t. It follows from (1.4) that the extra-stress T must be independent of time.

We assume that the body forces $\boldsymbol{g}$ (per unit mass) have a single-valued potential ψ:

$$\boldsymbol{g} = -\operatorname{grad}\psi. \tag{1.15}$$

We introduce the *modified pressure*:

$$\varphi = p + \varrho\psi. \tag{1.16}$$

It coincides with the ordinary pressure p if there are no body forces. The dynamical equations take the form

$$\operatorname{div} T - \operatorname{grad}\varphi = \varrho\dot{\boldsymbol{v}}. \tag{1.17}$$

2. A class of steady flows

We consider here steady flows for which the velocity field $\boldsymbol{v}(\boldsymbol{x})$ has the contravariant components

$$\{v^1, v^2, v^3\} = \{0, v(x^1), 0\} \tag{2.1}$$

in some appropriate *orthogonal* coordinate system x^1, x^2, x^3. The matrix $\|g_{ij}\|$ of the covariant components of the unit tensor I is then diagonal. We further assume that the g_{ij} are independent of x^2, so that $\|g_{ij}\|$ has the form

$$\|g_{ij}(\boldsymbol{x})\| = \begin{Vmatrix} [g_1(x^1, x^3)]^2 & 0 & 0 \\ 0 & [g_2(x^1, x^3)]^2 & 0 \\ 0 & 0 & [g_3(x^1, x^3)]^2 \end{Vmatrix}, \tag{2.2}$$

where g_1, g_2, and g_3 are three positive functions of x^1 and x^3. The flows defined in this manner are isochoric, because the equation

$$\operatorname{div}\boldsymbol{v} = 0, \quad \text{i.e.,} \quad (\sqrt{g})^{-1}\partial_i(\sqrt{g}\,v^i) = 0, \tag{2.3}$$

where $\sqrt{g} = g_1 g_2 g_3$, is obviously satisfied.

In the special case of the velocity field (2.1), integration of the differential equation (1.13) with the initial condition (1.14) yields the simple result

$$\xi^1 = x^1, \qquad \xi^2 = x^2 + s\,v(x^1), \qquad \xi^3 = x^3. \tag{2.4}$$

From (2.4), (1.12) and (1.11) we obtain

$$\|f^i_{(t)j}(t+s)\| = \begin{Vmatrix} 1 & 0 & 0 \\ s\,v'(x^1) & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{2.5}$$

where v' denotes the derivative of v. Equations (2.4) and (2.2) imply that

$$g_i(\xi^1, \xi^3) = g_i(x^1, x^3), \qquad i = 1, 2, 3; \tag{2.6}$$

i.e., that the covariant components of the unit tensor remain constant along the paths of the particles. Using equations (1.10), (2.5), (2.2), and (2.6) we obtain the following expression for the covariant components of the tensor $C_t^t(s)$:

$$\|c^{(t)}_{(t)ij}(s)\| = \left\| \begin{matrix} [g_1(x^1,x^3)]^2 + [s\,g_2(x^1,x^3)\,v'(x^1)]^2 & s\,[g_2(x^1,x^3)]^2\,v'(x^1) & 0 \\ s\,[g_2(x^1,x^3)]^2\,v'(x^1) & [g_2(x^1,x^3)]^2 & 0 \\ 0 & 0 & [g_3(x^1,x^3)]^2 \end{matrix} \right\|. \qquad (2.7)$$

The matrix $\big(C_t^t(s)\big)$ of the physical components of $C_t^t(s)$ is

$$\big(C_t^t(s)\big) = \left\| \frac{c^{(t)}_{(t)ij}(s)}{g_i(x^1,x^3)\,g_j(x^1,x^3)} \right\| = \left\| \begin{matrix} 1+\varkappa^2 s^2 & \varkappa s & 0 \\ \varkappa s & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right\|, \qquad (2.8)$$

where

$$\varkappa = \frac{g_2(x^1,x^3)}{g_1(x^1,x^3)}\,v'(x^1). \qquad (2.9)$$

In the remainder of this paper we shall always use the physical components of second order tensors.

3. Reduction of the general constitutive equation

The results of the previous section may be summarized in this way: If the velocity field has the form (2.1) in an orthogonal coordinate system obeying (2.2), the function C_t^t has the form

$$C_t^t(s) = I + s\,A + s^2\,B, \qquad (3.1)$$

where the matrices (A), (B) of the physical components of A and B are given by

$$(A) = \varkappa \left\| \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|, \qquad (B) = \varkappa^2 \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|. \qquad (3.2)$$

When (3.1) holds, the function C_t^t is completely determined by A and B, and the functional $\mathfrak{H}$ of (1.4) reduces to a function $\mathfrak{h}$ of the two tensors A and B, so that

$$T = \mathfrak{h}(A,B). \qquad (3.3)^\star$$

It follows from (1.2) that the tensor function $\mathfrak{h}$ must be isotropic; *i.e.*,

$$\mathfrak{h}(Q\,A\,Q^T,\,Q\,B\,Q^T) = Q\,\mathfrak{h}(A,B)\,Q^T \qquad (3.4)$$

must hold for all orthogonal tensors Q. It is a consequence of this fact that the component functions $\mathfrak{h}_{ij}$ of $\mathfrak{h}$ are the same for all *orthonormal* bases. Since physical components are components relative to orthonormal bases, the functions $\mathfrak{h}_{ij}$ do not depend on the coordinate system or the point under consideration, if physical components are used.

The normalization (1.6) of the functional $\mathfrak{H}$ implies that the function $\mathfrak{h}$ is normalized by

$$\operatorname{tr} T = \operatorname{tr}[\mathfrak{h}(A,B)] = 0. \qquad (3.5)$$

[★] We note that, if (3.1) holds, A and $2B$ coincide with the first two Rivlin-Ericksen tensors A_1 and A_2 as defined in [*1*], (8.9). Equation (3.3) has the form of a constitutive equation of the differential type (*cf.* [*1*] (23.14)).

Furthermore, it follows from (1.8) that

$$0 = \mathfrak{h}(0, 0). \tag{3.6}$$

We now consider the orthogonal tensor Q whose physical components are given by the matrix

$$(Q) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}. \tag{3.7}$$

Then, by (3.2), we have

$$Q A Q^T = A, \qquad Q B Q^T = B. \tag{3.8}$$

Hence, from (3.4) we get

$$Q T Q^T = T \tag{3.9}$$

for $T = \mathfrak{h}(A, B)$. Denoting the *physical components* of T by t_{ij}, we find that the matrix of the physical components of $Q T Q^T$ is given by

$$(Q T Q^T) = \begin{Vmatrix} t_{11} & t_{12} & -t_{13} \\ t_{21} & t_{22} & -t_{23} \\ -t_{31} & -t_{32} & t_{33} \end{Vmatrix}. \tag{3.10}$$

According to (3.9), this matrix must coincide with the matrix $(T) = \|t_{ij}\|$, which is possible only if $t_{13} = t_{31} = t_{23} = t_{32} = 0$. Hence (T) must have the form

$$(T) = \begin{Vmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{Vmatrix}. \tag{3.11}$$

By (3.5), we have

$$t_{11} + t_{22} + t_{33} = 0. \tag{3.12}$$

From the fact that the component functions $\mathfrak{h}_{ij}$ are independent of the coordinate system, and from (3.2) and (3.3), it follows that the physical components t_{ij} of T are functions of $\varkappa$ alone. It is clear from (3.11) and (3.12) that these components t_{ij} may be expressed in terms of three independent functions of $\varkappa$. We choose

$$t_{12} = \tau(\varkappa), \qquad t_{11} - t_{33} = \sigma_1(\varkappa), \qquad t_{22} - t_{33} = \sigma_2(\varkappa). \tag{3.13}$$

For the physical components $s_{ij} = t_{ij} - p\,\delta_{ij}$ of the stress tensor we also have

$$s_{12} = s_{21} = \tau(\varkappa), \tag{3.14}$$

$$s_{11} - s_{33} = \sigma_1(\varkappa), \qquad s_{22} - s_{33} = \sigma_2(\varkappa). \tag{3.15}$$

The three functions τ, σ_1, and σ_2 depend only on the material, not on the particular flow; they are *material functions.*

The functions τ, σ_1, and σ_2 are not completely arbitrary, as may be seen from the following argument: For a second choice of Q in (3.4) we take

$$(Q) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{3.16}$$

Equation (3.2) then yields

$$Q\,A\,Q^T = -A\,, \qquad Q\,B\,Q^T = B\,, \tag{3.17}$$

and, from (3.11), we get

$$(Q\,T\,Q^T) = \begin{Vmatrix} t_{11} & -t_{12} & 0 \\ -t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{Vmatrix} \tag{3.18}$$

It is clear from (3.2) that the conjugation operation (3.17) is equivalent to replacing $\varkappa$ by $-\varkappa$. Thus, by (3.13), equation (3.18) states that

$$\tau(-\varkappa) = -\tau(\varkappa)\,, \qquad \sigma_i(\varkappa) = \sigma_i(-\varkappa)\,; \tag{3.19}$$

i.e., that τ must be an *odd* function, while σ_1 and σ_2 must be *even* functions. No further restrictions are imposed on τ, σ_1, and σ_2 by the isotropy condition (3.4).

The condition (3.6) implies that the material functions τ, σ_1, and σ_2 must vanish for $\varkappa = 0$:

$$\tau(0) = \sigma_1(0) = \sigma_2(0) = 0\,. \tag{3.20}$$

A further restrictive condition on τ follows from the principle that the dissipation of energy must be positive. This condition is

$$\varkappa\,\tau(\varkappa) > 0\,, \qquad \varkappa \neq 0\,; \tag{3.21}$$

i.e. $\tau(\varkappa)$ must have the same sign as $\varkappa$. If it is assumed that τ is twice continuously differentiable, then (3.20) and (3.21) imply that $\tau(\varkappa)$ must be a strictly increasing function of $\varkappa$ in some interval $-\varkappa_0 \leqq \varkappa \leqq +\varkappa_0$ around $\varkappa = 0$. In this interval τ will have a strictly increasing and odd inverse, $\overset{-1}{\tau}$. In the remainder of this paper, whenever $\overset{-1}{\tau}$ occurs, we assume that we are in the range in which $\overset{-1}{\tau}$ is defined. If $\overset{-1}{\tau}$ is not defined over the entire real axis then the flows which we discuss below will not occur at high speeds.

We can give a precise meaning to the "shear-dependent viscosity" used in the rheological literature by identifying it with the material function η:

$$\eta(\varkappa) = \frac{\tau(\varkappa)}{\varkappa}\,. \tag{3.22}$$

Clearly η must be an *even* function. Since $\tau(0) = 0$ and τ is twice differentiable, η is differentiable, even at $\varkappa = 0$, and

$$\eta'(0) = 0\,. \tag{3.23}$$

If we assume that σ_1 and σ_2 are differentiable, we also have

$$\sigma_1'(0) = \sigma_2'(0) = 0\,. \tag{3.24}$$

If a polynomial approximation for η, σ_1, and σ_2, in a neighborhood of $\varkappa = 0$, is used to fit experimental data, only even powers of $\varkappa$ can occur.

4. Rectilinear flow

If the velocity field has the form (2.1) in a Cartesian coordinate system

$$x^1 = x\,, \qquad x^2 = y\,, \qquad x^3 = z\,, \tag{4.1}$$

then we call the motion *rectilinear flow*. In this case

$$g_1 = g_2 = g_3 = 1, \tag{4.2}$$

and hence, by (2.9),

$$\varkappa = v'(x). \tag{4.3}$$

By (3.13), the extra stresses t_{ij} are then functions of x only, and the dynamical equations (1.17) become

$$\partial_x t_{xx} - \partial_x \varphi = 0, \quad \partial_x t_{xy} - \partial_y \varphi = 0, \quad \partial_z \varphi = 0. \tag{4.4}$$

A simple analysis shows that these equations are satisfied only if

$$t_{xy}(x) = -a x + b, \tag{4.5}$$

$$\varphi = \varphi(x, y) = -a y + t_{xx}(x) - c, \tag{4.6}$$

where a, b, and c are constants.

There are two important special cases of rectilinear flow which we shall discuss in the next two sections.

5. Simple shearing flow

This is a rectilinear flow between an infinite plate I at rest and an infinite plate II moving with a constant speed V parallel to plate I. We denote the distance between the plates by d. We choose the y and z axes to be in the plane of plate I and such that plate II moves in the y direction. The x axis is then perpendicular to the plates with $x=0$ at plate I. Assuming that the fluid adheres to the walls, we have the boundary conditions

$$v(0) = 0, \quad v(d) = V. \tag{5.1}$$

We assume now that the modified pressure φ, defined by (1.16), has no gradient in direction of the flow; *i.e.*, that $\partial_y \varphi = 0$. We then have $a=0$ in (4.6). Hence, by (3.13), (4.3) and (4.5), the shearing stress

$$t_{xy}(x) = \tau(v'(x)) = b \tag{5.2}$$

is constant. It follows that $v'(x) = \overset{-1}{\tau}(b)$ must be constant and hence that $v(x)$ is linear. The conditions (5.1) then imply that

$$v(x) = \frac{V}{d} x, \tag{5.3}$$

and (4.3) becomes

$$\varkappa = \frac{V}{d} = \text{const}. \tag{5.4}$$

According to (3.14), the shearing stress is

$$s_{xy} = \tau\left(\frac{V}{d}\right). \tag{5.5}$$

Since s_{xy} is equal to the tangential force per unit area that must be applied to the moving plate II in order to produce the flow, equation (5.5) can be used to determine the material function τ experimentally.

It follows from (5.4) and (3.13) that the t_{ij} must actually be constant. Hence, by (4.6), since $a=0$,

$$\varphi = t_{xx} - c = \text{const}. \tag{5.6}$$

Using (1.16) and $s_{xx} = t_{xx} - p$, we find

$$s_{xx} = c - \varrho\psi, \tag{5.7}$$

and (3.15) becomes

$$s_{xx} - s_{zz} = \sigma_1\left(\frac{V}{d}\right), \qquad s_{yy} - s_{zz} = \sigma_2\left(\frac{V}{d}\right). \tag{5.8}$$

Hence, the matrix of the stress tensor is

$$(S) = \left\| \begin{matrix} c - \varrho\psi & \tau\left(\frac{V}{d}\right) & 0 \\ \tau\left(\frac{V}{d}\right) & c - \varrho\psi + \sigma_2\left(\frac{V}{d}\right) - \sigma_1\left(\frac{V}{d}\right) & 0 \\ 0 & 0 & c - \varrho\psi - \sigma_1\left(\frac{V}{d}\right) \end{matrix} \right\|. \tag{5.9}$$

Since the normal stresses can be measured, at least in principle, equations (5.8) can be used to determine the material functions σ_1 and σ_2.

6. Flow through a channel

This is a rectilinear flow between two parallel infinite plates which are both at rest. We now place the y and z axes in a plane half-way between the plates; the x axis is then perpendicular to the plates. Let the distance between the plates be d; it follows that the plates are in the planes $x = \pm\frac{1}{2}d$.

The fluid adheres to the plates. For rectilinear flow, to get any motion at all we must have $a \neq 0$ in (4.6). When a is positive, there is a flow in the positive y-direction. From (3.13), (4.3) and (4.5) we get $v'(x) = \overset{-1}{\tau}(-a\,x + b)$. The boundary conditions are $v(\pm\frac{1}{2}d) = 0$. The fact that $\overset{-1}{\tau}$ is an odd function implies $b=0$; *i.e.*,

$$v'(x) = -\overset{-1}{\tau}(a\,x). \tag{6.1}$$

After integration we obtain the velocity profile

$$v(x) = -\int_{-\frac{1}{2}d}^{x} \overset{-1}{\tau}(a\,\xi)\,d\xi. \tag{6.2}$$

The total applied force f in the direction of the flow, exerted on a column of fluid with cross-section $\mathscr{A}$ and lying between two planes $y = y_{\mathrm{I}}$ and $y = y_{\mathrm{II}}$, is given by

$$f = \Bigl(\int_{\mathscr{A}} s_{yy}\,dA\Bigr)_{y=y_{\mathrm{II}}} - \Bigl(\int_{\mathscr{A}} s_{yy}\,dA\Bigr)_{y=y_{\mathrm{I}}} - \int_{y_{\mathrm{I}}}^{y_{\mathrm{II}}} \int_{\mathscr{A}} (\partial_y\psi)\,\varrho\,dA\,dy. \tag{6.3}$$

This reduces to

$$f = \int_{\mathscr{A}} \{(s_{yy} - \varrho\psi)_{y=y_{\mathrm{II}}} - (s_{yy} - \varrho\psi)_{y=y_{\mathrm{I}}}\}\,dA. \tag{6.4}$$

By (1.16) and (4.6) we have

$$s_{yy} - \varrho\psi = t_{yy} - p - \varrho\psi = t_{yy} - \varphi = a\,y + t_{yy}(x) - t_{xx}(x) + c,$$

hence

$$s_{yy} - \varrho\psi = a\,y + h(x)\,. \tag{6.5}$$

Substitution into (6.4) yields

$$f = a\,(y_{\mathrm{II}} - y_{\mathrm{I}})\,A\,,$$

where A is the area of the cross-section of the column. We thus have

$$a = \frac{f}{A\,(y_{\mathrm{II}} - y_{\mathrm{I}})}\,, \tag{6.6}$$

which gives a physical meaning to the parameter a. It is the *applied force per unit volume in the direction of the flow*. It is the driving force that produces the flow. If there are no body forces, a reduces to what is usually called the *pressure head* per unit length. It must be produced by a pump or similar device. If the only body force is gravity, acting in the direction of flow, and if there is no pressure head, then a is simply the *specific weight* of the fluid. In any case, a is a measurable quantity. We call it the *driving force*.

The volume discharge per unit time through a cross-section of the channel one unit deep is

$$Q = \int\limits_{-\frac{1}{2}d}^{\frac{1}{2}d} v(x)\,d\,x\,. \tag{6.7}$$

Integrating by parts, remembering the boundary conditions $v(\pm\frac{1}{2}d) = 0$, and using (6.1), we find

$$Q = -\int\limits_{\frac{1}{2}d}^{\frac{1}{2}d} x\,v'(x)\,d\,x = \int\limits_{-\frac{1}{2}d}^{+\frac{1}{2}d} x\,\overset{-1}{\tau}(a\,x)\,d\,x\,. \tag{6.8}$$

We finally obtain

$$Q = \frac{2}{a^2}\int\limits_{0}^{\frac{1}{2}a\,d} \xi\,\overset{-1}{\tau}(\xi)\,d\,\xi\,. \tag{6.9}$$

Thus, the experimentally measurable rate of discharge Q is related to the measurable driving force a through the material function τ which can be determined by independent measurements.

Equation (6.9) is easily solved for the function $\overset{-1}{\tau}$ in terms of the function $Q = Q(a)$:

$$\overset{-1}{\tau}\left(\frac{a\,d}{2}\right) = \frac{2}{a\,d^2}\,\frac{\mathrm{d}}{\mathrm{d}\,a}\left[a^2\,Q(a)\right]. \tag{6.10}$$

This formula may be used to calculate the material function τ when Q has been measured as a function of a.

The normal stresses are obtained from (4.6), (3.15), (4.3) and (6.1):

$$\begin{aligned} s_{xx} &= a\,y + \varrho\psi - c\,,\\ s_{xx} - s_{zz} &= \sigma_1\big(\overset{-1}{\tau}(a\,x)\big)\,,\\ s_{yy} - s_{zz} &= \sigma_2\big(\overset{-1}{\tau}(a\,x)\big)\,. \end{aligned} \tag{6.11}$$

If the material functions τ, σ_1, and σ_2 are known from independent measurements, these formulae may be used to predict the measurable normal stresses. It will be noticed that the normal stresses are determined only up to a constant hydrostatic pressure of magnitude c.

If only τ is known, then (6.11) may be used to determine σ_1 and σ_2 by measuring $s_{xx}-s_{zz}$ and $s_{yy}-s_{zz}$. If the potential ψ of the body forces is known, then a measurement of s_{xx} just determines the value of the constant c.

7. Poiseuille flow

We consider a steady flow through an infinite circular pipe of radius R and use cylindrical coordinates with the z-axis coincident with the axis of the pipe. We label the coordinates as follows:

$$r = x^1, \qquad z = x^2, \qquad \vartheta = x^3. \tag{7.1}$$

We say that we have Poiseuille flow if in this coordinate system the velocity field has the form (2.1) and if the fluid adheres to the walls of the pipe; *i.e.*

$$v(R) = 0. \tag{7.2}$$

The coordinate system r, z, ϑ is orthogonal and obeys (2.2) with

$$g_1 = g_2 = 1, \qquad g_3 = r. \tag{7.3}$$

Therefore, we find that (2.9) becomes

$$\varkappa = v'(r). \tag{7.4}$$

By (3.13), the extra stresses t_{ij} are then functions of r only, and the dynamical equations (1.17) become

$$\begin{aligned} \partial_r t_{rr} + r^{-1}(t_{rr} - t_{\vartheta\vartheta}) - \partial_r \varphi &= 0, \\ \partial_r t_{rz} + r^{-1} t_{rz} - \partial_z \varphi &= 0, \\ - \partial_\vartheta \varphi &= 0. \end{aligned} \tag{7.5}$$

An elementary analysis shows that these equations are satisfied only if

$$t_{rz} = -\tfrac{1}{2} a r + b r^{-1}, \tag{7.6}$$

$$\varphi = \varphi(r,z) = -a z + h(r), \tag{7.7}$$

$$h'(r) = \partial_r t_{rr} + r^{-1}(t_{rr} - t_{\vartheta\vartheta}), \tag{7.8}$$

where a and b are constants and h is an arbitrary function of r. Since the shearing stress $s_{rz} = t_{rz}$ must be continuous at the center of the pipe (*i.e.* for $r=0$), we must put $b=0$ in (7.6). According to (3.14) and (7.4), we thus have

$$s_{rz} = -\tfrac{1}{2} a r = \tau\big(v'(r)\big);$$

hence, since τ is an odd function,

$$v'(r) = -\overset{-1}{\tau}(\tfrac{1}{2} a r). \tag{7.9}$$

This equation, together with the boundary condition (7.2), determines the velocity profile:

$$v(r) = \int_r^R \overset{-1}{\tau}(\tfrac{1}{2} a \xi)\, d\xi. \tag{7.10}$$

An investigation analogous to that given in § 6 for the case of a flow through a channel shows that the parameter a is again the applied force per unit volume in the direction of the flow. We again call a the *driving force*. It reduces to the pressure head per unit length if there are no body forces. If the flow takes place in a vertical pipe under the action of gravity alone, a is just the specific weight of the fluid.

The volume discharge per unit time through a cross-section of the pipe is

$$Q = 2\pi \int_0^R v(r)\, r\, dr. \tag{7.11}$$

After integration by parts and use of (7.2) and (7.9), we find*

$$Q = \frac{8\pi}{a^3} \int_0^{\frac{1}{2}aR} \xi^2 \overset{-1}{\tau}(\xi)\, d\xi. \tag{7.12}$$

Thus, if the material function τ is known, one can predict, for pipe flow, the dependence of the rate of discharge Q on the driving force a.

As in the case of channel flow, (7.12) is easily inverted:

$$\overset{-1}{\tau}\left(\tfrac{1}{2} a R\right) = \frac{1}{\pi a^2 R^3} \frac{\mathrm{d}}{\mathrm{d}a} \left[a^3 Q(a)\right]. \tag{7.13}$$

If $Q = Q(a)$ has been measured, this formula may be used to calculate the material function τ.

To calculate the normal stresses, we observe that (3.13), (7.4), (7.9) and the fact that σ_1 is an even function imply

$$t_{rr} - t_{\vartheta\vartheta} = \sigma_1\left(\overset{-1}{\tau}\left(\tfrac{1}{2} a r\right)\right). \tag{7.14}$$

Integration of (7.8) yields

$$h(r) = t_{rr} - \int_r^R \xi^{-1} \sigma_1\left(\overset{-1}{\tau}\left(\tfrac{1}{2} a \xi\right)\right) d\xi - c, \tag{7.15}$$

where c is a constant. Substituting (7.15) into (7.7) and using $\varphi = p + \varrho\psi$ and $s_{rr} = t_{rr} - p$, we obtain

$$s_{rr} = a z + \varrho\psi + \int_r^R \xi^{-1} \sigma_1\left(\overset{-1}{\tau}\left(\tfrac{1}{2} a \xi\right)\right) d\xi + c. \tag{7.16}$$

By (3.15), we have

$$s_{rr} - s_{\vartheta\vartheta} = \sigma_1\left(\overset{-1}{\tau}\left(\tfrac{1}{2} a r\right)\right), \qquad s_{zz} - s_{\vartheta\vartheta} = \sigma_2\left(\overset{-1}{\tau}\left(\tfrac{1}{2} a r\right)\right). \tag{7.17}$$

The remarks made in § 6 about the prediction of measurable normal stresses in channel flow also apply here.

8. Couette flow

Here we consider a steady flow between two infinite coaxial cylinders; the inner cylinder has radius R_1, and the outer cylinder has radius R_2. Using cylindrical coordinates with the z axis coincident with the axis of the cylinders, we now label the coordinates

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* Equivalent relations have appeared previously in the literature (*cf.* [5], pp. 249 and 250). The derivation given here, we believe, is the first that is not based on special assumptions.

$$r = x^1, \qquad \vartheta = x^2, \qquad z = x^3. \tag{8.1*}$$

If, in these coordinates, the velocity field has the form (2.1), we speak about Couette flow. We than have

$$g_1 = g_3 = 1, \qquad g_2 = r, \tag{8.2}$$

so that (2.9) becomes

$$\varkappa = r\, v'(r). \tag{8.3}$$

Note that $v(r)$ has now the physical dimension of an angular velocity. By (3.13) and (8.3), the extra stresses t_{ij} depend only on r. The dynamical equations (1.17) become

$$\begin{aligned} \partial_r t_{rr} + r^{-1}(t_{rr} - t_{\vartheta\vartheta}) - \partial_r \varphi &= -\varrho r v^2, \\ \partial_r t_{r\vartheta} + 2r^{-1} t_{r\vartheta} - r^{-1}\partial_\vartheta \varphi &= 0, \\ -\partial_z \varphi &= 0. \end{aligned} \tag{8.4}$$

An analysis of these equations, using the fact that φ must be single valued and hence periodic in ϑ, shows that φ can depend only on r and that $t_{r\vartheta} = M/(2\pi r^2)$, where M is a constant. From (3.13) and (8.3), it follows that

$$t_{r\vartheta} = \tau\big(r\, v'(r)\big) = \frac{M}{2\pi r^2}. \tag{8.5}$$

The moment per unit height exerted on the fluid inside the cylindrical surface $r = \text{const.}$ is, by (8.5), equal to

$$r(2\pi r)\, t_{r\vartheta} = M \tag{8.6}$$

and is independent of r. From (8.5) we get

$$v'(r) = \frac{1}{r}\, \overset{-1}{\tau}\left(\frac{M}{2\pi r^2}\right). \tag{8.7}$$

We consider further the special case in which the inner cylinder remains at rest while the outer cylinder rotates with constant angular velocity Ω. Assuming that the fluid adheres to the walls, we then have

$$v(R_1) = 0, \qquad v(R_2) = \Omega. \tag{8.8}$$

M is the moment per unit height that must be applied to the outer cylinder in order to produce the flow.

Integration of (8.7) with the boundary condition (8.8) yields

$$\Omega = \frac{1}{2} \int_{M/(2\pi R_2^2)}^{M/(2\pi R_1^2)} \frac{1}{\xi}\, \overset{-1}{\tau}(\xi)\, d\xi. \tag{8.9**}$$

Thus, a knowledge of the material function τ makes it possible to predict the angular velocity Ω of the outer cylinder as a function of the moment per unit height M. If the gap between the two cylinders is very small $[(R_2 - R_1)/R_1 \ll 1]$, the formula (8.9) may be approximated by

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* Note that this differs from the labeling (7.1) used for Poiseuille flow.

** A derivation, based on special assumptions, of a formula equivalent to (8.9) is given in [6], § 3.

$$\Omega \approx \frac{R_2 - R_1}{R_1} \overset{-1}{\tau}\left(\frac{M}{2\pi R_2^2}\right). \tag{8.10}$$

We wish to invert (8.9) so as to be able to calculate the material function τ from an experimental determination of $\Omega = \Omega(M)$. Differentiation of (8.9) with respect to M gives

$$2M \frac{\mathrm{d}\Omega(M)}{\mathrm{d}M} = \overset{-1}{\tau}\left(\frac{M}{2\pi R_1^2}\right) - \overset{-1}{\tau}\left(\frac{M}{2\pi R_2^2}\right). \tag{8.11}$$

With the abbreviations

$$F(M) = 2M \frac{\mathrm{d}\Omega(M)}{\mathrm{d}M},, \quad \alpha = \frac{R_1^2}{R_2^2}, \tag{8.12}$$

equation (8.11) becomes

$$F(M) = \overset{-1}{\tau}\left(\frac{M^M}{2\pi R_1^2}\right) - \overset{-1}{\tau}\left(\alpha \frac{M}{2\pi R_1^2}\right). \tag{8.13}$$

If we replace here M by $\alpha^n M$ and sum from $n=0$ to $n=N$, we get

$$\sum_{n=0}^{N} F(\alpha^n M) = \overset{-1}{\tau}\left(\frac{M}{2\pi R_1^2}\right) - \overset{-1}{\tau}\left(\alpha^{n+1} \frac{M}{2\pi R_1^2}\right). \tag{8.14}$$

Since $R_1 < R_2$, we have $\alpha < 1$ and hence $\lim_{n\to\infty} \alpha^{n+1} = 0$. The continuity of $\overset{-1}{\tau}$ at zero and the fact that $\overset{-1}{\tau}(0) = 0$ (*cf.* (3.20)) insure that

$$\lim_{n\to\infty} \overset{-1}{\tau}\left(\alpha^{n+1} \frac{M}{2\pi R_1^2}\right) = 0.$$

Hence, (8.14) yields*

$$\overset{-1}{\tau}\left(\frac{M}{2\pi R_1^2}\right) = \sum_{n=0}^{\infty} F(\alpha^n M). \tag{8.15}$$

Equations (8.12) and (8.15) make it possible to calculate τ if $\Omega = \Omega(M)$ is known. When the gap between the cylinders is small, τ can be calculated directly from (8.10).

We now determine the normal stresses. From (3.13), (8.3), and (8.7), we get

$$t_{rr} - t_{\vartheta\vartheta} = \sigma_1\left(\overset{-1}{\tau}\left(\frac{M}{2\pi r^2}\right)\right) - \sigma_2\left(\overset{-1}{\tau}\left(\frac{M}{2\pi r^2}\right)\right). \tag{8.16}$$

Substitution of (8.16) into the first equation of (8.4) and integration yields, after using $s_{rr} = t_{rr} - p$ and $\varphi = p + \varrho\psi$,

$$s_{rr} = \varrho\psi - \int_{R_1}^{r} \left\{\varrho\xi[v(\xi)]^2 + \frac{1}{\xi}\sigma_1\left(\overset{-1}{\tau}\left(\frac{M}{2\pi\xi^2}\right)\right) - \frac{1}{\xi}\sigma_2\left(\overset{-1}{\tau}\left(\frac{M}{2\pi\xi^2}\right)\right)\right\} d\xi + c, \tag{8.17}$$

where $v(\xi)$, by (8.7) and (8.8), is

$$v(\xi) = \int_{R_1}^{\xi} \frac{1}{\zeta} \overset{-1}{\tau}\left(\frac{M}{2\pi\zeta^2}\right) d\zeta. \tag{8.18}$$

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* Note added in proof: It has been brought to our attention that an equivalent inversion of (8.9) was published by Krieger & Elrod [7].

By (3.15), we have

$$s_{rr} - s_{zz} = \sigma_1\left(\overset{-1}{\tau}\left(\frac{M}{2\pi r^2}\right)\right), \qquad s_{\vartheta\vartheta} - s_{zz} = \sigma_2\left(\overset{-1}{\tau}\left(\frac{M}{2\pi r^2}\right)\right), \tag{8.19}$$

The remarks made at the end of § 6 apply here also.

When the potential ψ of the body forces is independent of r, (8.17) yields the following expression for the difference of the normal stresses at the outer and inner cylinders:

$$s_{rr}(R_2) - s_{rr}(R_1) = -\int\limits_{R_1}^{R_2} \left\{\varrho\, r\,[v(r)]^2 + \frac{1}{r}\,\sigma_1\left(\overset{-1}{\tau}\left(\frac{M}{2\pi r^2}\right)\right) - \frac{1}{r}\,\sigma_2\left(\overset{-1}{\tau}\left(\frac{M}{2\pi r^2}\right)\right)\right\} dr. \tag{8.20}$$

When $(R_2 - R_1)/R_1 \ll 1$, this formula may be approximated by

$$s_{rr}(R_2) - s_{rr}(R_1) \approx \frac{R_2 - R_1}{R_1}\left[\sigma_2\left(\overset{-1}{\tau}\left(\frac{M}{2\pi R_2^2}\right)\right) - \sigma_1\left(\overset{-1}{\tau}\left(\frac{M}{2\pi R_2^2}\right)\right)\right]. \tag{8.21}$$

The error in both the approximate formulae (8.10) and (8.21) is of the order of $[(R_2 - R_1)/R_1]^2$. The inertial term $\varrho r [v(r)]^2$ in the integral on the right side of (8.20) makes a contribution to $s_{rr}(R_2) - s_{rr}(R_1)$ of the order of $[(R_2 - R_1)/R_1]^3$ and hence does not appear in (8.21).

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The Constitutive Equations for Certain Classes of Deformations

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1. Introduction

In previous papers [*1, 2*] constitutive equations have been developed for a certain class of deformations in materials for which the stress at any instant of time depends, in general, not only on the displacement gradients at that instant of time, but also on the previous deformation history of the material.

We considered [*1*] that a material is undeformed until time zero and that, in the short time interval 0 to t_1, it is subjected to a deformation which is thereafter maintained constant. It was assumed that the material is such that for deformations carried out in a short time t_1, the material behaves substantially as a perfectly-elastic material, but for deformations carried out during much longer times the stress depends on the deformation history up to the instant considered. For the limited class of deformations which are maintained constant after time t_1, it is apparent that the stress at an instant of time $t\,(\gg t_1)$ will depend only on the (constant) displacement gradients and on the time t. With this as starting point, it was shown that constitutive equations closely related to those which are valid in finite elasticity theory can be obtained.

For example, in the case when the material is incompressible and isotropic in its undeformed state, the constitutive equation takes the form

$$t_{ij} = -p\,\delta_{ij} + \alpha_1 g_{ij} + \alpha_2 g_{ik} g_{kj}. \tag{1.1}$$

Here g_{ij} is defined by

$$g_{ij} = \frac{\partial x_i}{\partial X_k}\,\frac{\partial x_j}{\partial X_k}. \tag{1.2}$$

x_i and X_i are the coordinates in a rectangular Cartesian coordinate system x of a generic particle of the body in the (constant) deformed state and in the undeformed state respectively and t_{ij} are the stress components in the coordinate system x. In equation (1.1), p is an arbitrary hydrostatic pressure and α_1 and α_2 are functions of t and of I_1 and I_2 defined by

$$I_1 = g_{ii} \quad \text{and} \quad I_2 = \tfrac{1}{2}\,(g_{ii} g_{jj} - g_{ij} g_{ji}). \tag{1.3}$$

It was then shown that formulae can be derived for the forces necessary to maintain certain simple physically important types of deformation of the class considered, without making further assumptions regarding the manner in which α_1 and α_2 depend on I_1, I_2 and t.

In a subsequent paper [2] it was pointed out that the assumption that the material behaves as a perfectly-elastic material for deformations carried out during short times t_1 is an unnecessarily restrictive one for the validity of equation (1.1) as the constitutive equation for deformations of the class under consideration in an incompressible, isotropic material. We may instead merely assume that the stress at time t has a "memory" of the deformation to which the material was subjected at earlier times which decays sufficiently rapidly.

It was also shown [2] that if the gradients of the (constant) displacement u_i are small compared with unity, (1.1) may be re-written as

$$t_{ij} = -p\,\delta_{ij} + \beta_1 e_{ij} + \beta_2 e_{ik} e_{kj}, \tag{1.4}$$

where e_{ij} is defined by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{1.5}$$

and β_1 and β_2 are functions of t and of J_1 and J_2 defined by

$$J_1 = \tfrac{1}{2}(e_{ij} e_{ji} - e_{ii} e_{jj}) \quad \text{and} \quad J_2 = |e_{ij}|. \tag{1.6}$$

Furthermore, the validity of such equations for a certain class of materials was demonstrated. It was found that for these materials $\beta_2 = 0$ and β_1 took the form

$$\beta_1 = \varphi(t)\,\gamma(J_1). \tag{1.7}$$

In the present paper, we shall show that constitutive equations which are formally similar to (1.4) may be derived on the basis of the assumption that the displacement components in the coordinate system x all vary with time according to a specified law which is the same for each of the displacement components and independent of the particle considered (see equation (2.3) below). The forces which must be applied to the ends of a rod of circular cross-section in order to produce in it simultaneously a torsion and simple extension for which the amount of torsion and fractional extension vary with time in the same manner are then calculated. Finally, we discuss the case when the time dependence of the constitutive equation occurs as a common multiplier of all the terms in it.

2. Basic Assumptions

We consider a body to undergo a deformation such that a particle of the body which is initially (at time $\tau = 0$) at X_i in the rectangular Cartesian coordinate system x moves at time τ to x_i in the same coordinate system. The displacement components u_i in the coordinate system x are defined by

$$u_i = x_i - X_i \tag{2.1}$$

and the deformation is completely specified if the functional dependence of u_i on X_i and τ is given by a relation of the type

$$u_i = u_i(X_j, \tau). \tag{2.2}$$

Now, let us consider the class of deformations for which the relation (2.2) has the form

$$u_i = f(\tau)\, v_i(X_j), \tag{2.3}$$

where $f(\tau)$ is some specified function of τ and v_i is a function of X_j only.

We shall now assume that once we have specified the function $f(\tau)$, the components of stress t_{ij} at time t in the system x at a point of the body, which was at X_i at time $\tau=0$, depend only on the gradients $\partial v_i/\partial X_j$ at the point; *i.e.*

$$t_{ij}=\varphi_{ij}(\partial v_p/\partial X_q, t), \tag{2.4}$$

where φ_{ij} is a single-valued function of the arguments $\partial v_p/\partial X_q$ and t, which is completely determined by $f(\tau)$.

We now assume that $\partial u_i/\partial X_j \ll 1$. It is then possible to impose certain restrictions on the manner in which φ_{ij} can depend on $\partial v_p/\partial X_q$. These are discussed in Section 3.

3. Derivation of the Constitutive Equation

Let us now consider that the body undergoes, in addition to the displacements defined by (2.3), small rigid rotations of amounts $f(\tau)\Omega_i$ about the axes x_i. Let $\bar{u}_i$ be the displacement components of a generic particle of the body, initially at x_i, in these rotations. Then, provided the rotations are sufficiently small so that $\partial \bar{u}_i/\partial x_j \ll 1$, we have

$$\bar{u}_i=f(\tau)\,\varepsilon_{ijk}\Omega_j x_k. \tag{3.1}$$

Since we have already assumed $\partial u_i/\partial X_j \ll 1$, it follows, with (2.1) ,that $\partial \bar{u}_i/\partial X_j \ll 1$. Neglecting terms of the second degree in $\partial u_i/\partial X_j$ and $\partial \bar{u}_i/\partial X_j$ in comparison with those of the first degree, and employing the notation

$$\bar{v}_i=\varepsilon_{ijk}\Omega_j X_k, \tag{3.2}$$

we have

$$\frac{\partial}{\partial X_j}(u_i+\bar{u}_i)=f(\tau)\left[\frac{\partial v_i}{\partial X_j}+\frac{\partial \bar{v}_i}{\partial X_j}\right]. \tag{3.3}$$

With the same approximation, we may take the stress components at time t in the coordinate system x associated with the displacement field $u_i+\bar{u}_i$ to be the same those as associated with the displacement field u_i. We thus have, with (2.4),

$$\varphi_{ij}\left(\frac{\partial v_p}{\partial X_q}, t\right)=\varphi_{ij}\left(\frac{\partial v_p}{\partial X_q}+\frac{\partial \bar{v}_p}{\partial X_q}, t\right), \tag{3.4}$$

for all $\bar{v}_i$ of the form (3.2), *i.e.* for all $\partial \bar{v}_p/\partial X_q$ of the form

$$\frac{\partial \bar{v}_p}{\partial X_q}=-\varepsilon_{pqr}\Omega_r. \tag{3.5}$$

Taking

$$\Omega_r=\frac{1}{2}\varepsilon_{rmn}\frac{\partial v_m}{\partial X_n}, \tag{3.6}$$

we obtain, from (3.5),

$$\frac{\partial \bar{v}_p}{\partial X_q}=-\frac{1}{2}\varepsilon_{rmn}\,\varepsilon_{rpq}\frac{\partial v_m}{\partial X_n}. \tag{3.7}$$

Since

$$\varepsilon_{rmn}\,\varepsilon_{rpq}=\delta_{mp}\,\delta_{nq}-\delta_{mq}\,\delta_{np} \tag{3.8}$$

equation (3.7) yields

$$\frac{\partial \bar{v}_p}{\partial X_q}=\frac{1}{2}\left(\frac{\partial v_q}{\partial X_p}-\frac{\partial v_p}{\partial X_q}\right). \tag{3.9}$$

From (3.9) and (3.4), we obtain

$$\varphi_{ij}\left(\frac{\partial v_p}{\partial X_q}, t\right) = \varphi_{ij}\left[\frac{1}{2}\left(\frac{\partial v_p}{\partial X_q} + \frac{\partial v_q}{\partial X_p}\right), t\right]. \tag{3.10}$$

Since

$$\frac{\partial \bar{v}_p}{\partial X_q} + \frac{\partial \bar{v}_q}{\partial X_p} = 0 \tag{3.11}$$

for all $\partial \bar{v}_p / \partial X_q$ of the form (3.5), the validity of the relation (3.10) implies the validity of the relation (3.4) for all $\partial \bar{v}_p / \partial X_q$ of the form (3.5). We thus have the result, from (2.4), that

$$t_{ij} = \varphi_{ij}(e_{pq}, t), \tag{3.12}$$

where e_{pq} is defined by

$$e_{pq} = \frac{1}{2}\left(\frac{\partial v_p}{\partial X_q} + \frac{\partial v_q}{\partial X_p}\right). \tag{3.13}$$

4. The Isotropic Case

We now consider that the material of the body is isotropic at time $\tau = 0$. Let x' be a generic rectangular Cartesian coordinate system related to the system x by

$$x'_i = a_{ij} x_j, \tag{4.1}$$

where a_{ij} are constants satisfying the orthogonality conditions

$$a_{ij} a_{ik} = a_{ji} a_{ki} = \delta_{jk}. \tag{4.2}$$

Let X'_i be the coordinates in the system x' of the point which is at X_i in the system x. Let t'_{ij} and u'_i be the stress components and displacement components respectively in the system x'. Then, we have

$$X'_i = a_{ip} X_p, \quad u'_i = a_{ip} u_p \quad \text{and} \quad t'_{ij} = a_{ip} a_{jq} t_{pq}. \tag{4.3}$$

Employing the notation

$$u'_i = f(\tau)\, v'_i,$$

we obtain from (2.3) and the second of equations (4.3)

$$v'_i = a_{ip} v_p. \tag{4.4}$$

Defining e'_{pq} by

$$e'_{pq} = \frac{1}{2}\left(\frac{\partial v'_p}{\partial X'_q} + \frac{\partial v'_q}{\partial X'_p}\right), \tag{4.5}$$

we see that

$$e'_{ij} = a_{ip} a_{jq} e_{pq}. \tag{4.6}$$

It follows from (3.12) and the fact that the material is isotropic at time $\tau = 0$ that

$$t'_{ij} = \varphi_{ij}(e'_{pq}, t). \tag{4.7}$$

From (3.12), (4.3) and (4.7), we obtain

$$a_{im} a_{jn} \varphi_{mn}(e_{pq}, t) = \varphi_{ij}(e'_{pq}, t), \tag{4.8}$$

where e'_{pq} is given by (4.6), for all a_{ij} satisfying (4.2). This equation represents the limitation imposed on the form of φ_{ij} by isotropy of the material when $\tau = 0$.

Let ψ_{ij} and ψ'_{ij} be the components in the coordinate systems x and $\bar{x}$ respectively of an arbitrary symmetric Cartesian tensor of the second order. Then,

$$\psi_{mn} = a_{im}\, a_{jn}\, \psi'_{ij}. \tag{4.9}$$

Multiplying the relation (4.8) throughout by ψ'_{ij}, and employing the relation (4.9), we obtain

$$\psi_{ij}\, \varphi_{ij}(e_{pq}, t) = \psi'_{ij}\, \varphi_{ij}(e'_{pq}, t). \tag{4.10}$$

With the notation $F = \psi_{ij}\, \varphi_{ij}(e_{pq}, t)$, we see that

$$\varphi_{ij} = \frac{1}{2}\left(\frac{\partial F}{\partial \psi_{ij}} + \frac{\partial F}{\partial \psi_{ji}}\right). \tag{4.11}$$

The relation (4.10) expresses the scalar invariance of F under orthogonal transformations.

If we assume initially in (2.4) that φ_{ij} is a polynomial in $\partial v_p/\partial X_q$, then in (4.10) φ_{ij} is a polynomial in e_{pq}. F is then a polynomial in e_{pq} and ψ_{pq}, linear in the latter. F must therefore be expressible as a polynomial in the elements of an integrity basis under the orthogonal group for the two tensors ψ_{ij} and e_{ij}. With the notation

$$\boldsymbol{\psi} = \|\psi_{ij}\| \quad \text{and} \quad \boldsymbol{e} = \|e_{ij}\|, \tag{4.12}$$

we thus obtain

$$\boldsymbol{F} = \alpha_0 \operatorname{tr} \boldsymbol{\psi} + \alpha_1 \operatorname{tr} \boldsymbol{\psi}\boldsymbol{e} + \alpha_2 \operatorname{tr} \boldsymbol{\psi}\boldsymbol{e}^2, \tag{4.13}$$

where α_0, α_1 and α_2 are polynomials in $\operatorname{tr}\boldsymbol{e}$, $\operatorname{tr}\boldsymbol{e}^2$ and $\operatorname{tr}\boldsymbol{e}^3$ and are single-valued functions of t. From (4.13), (4.11) and (4.12), we readily obtain

$$t_{ij} = \alpha_0\, \delta_{ij} + \alpha_1\, e_{ij} + \alpha_2\, e_{ik}\, e_{kj}. \tag{4.14}$$

If φ_{ij} is a single-valued function of e_{pq}, then we can employ a procedure used, in a somewhat different context, by Ericksen & Rivlin [*3*] to show that an equation of the form (4.14) is still valid with α_0, α_1 and α_2 single-valued functions of $\operatorname{tr}\boldsymbol{e}$, $\operatorname{tr}\boldsymbol{e}^2$, $\operatorname{tr}\boldsymbol{e}^3$ and t.

5. The Incompressible Case

If the material considered is incompressible, we have, still assuming $\partial u_i/\partial X_j \ll 1$,

$$\frac{\partial u_i}{\partial X_i} = 0. \tag{5.1}$$

With (2.3) and (3.13), this equation may be rewritten as

$$\operatorname{tr}\boldsymbol{e} = 0. \tag{5.2}$$

Also, noting that if the deformation is specified the stress is undetermined to the extent of an arbitrary hydrostatic pressure $-p$ (say), we replace our initial assumption (2.4) by the relation

$$t_{ij} = \varphi_{ij}\left(\frac{\partial x_p}{\partial X_q}, t\right) - p\, \delta_{ij}. \tag{5.3}$$

We then obtain in a manner similar to that adopted in Sections 3 and 4 that, in general,

$$t_{ij} = \varphi_{ij}(e_{pq}, t) - p\, \delta_{ij} \tag{5.4}$$

and, if the material is isotropic at time $\tau=0$,

$$t_{ij}=-p\,\delta_{ij}+\alpha_1 e_{ij}+\alpha_2 e_{ik} e_{kj}, \tag{5.5}$$

where α_1 and α_2 are single-valued functions of $\operatorname{tr}\boldsymbol{e}^2$, $\operatorname{tr}\boldsymbol{e}^3$ and t, and are polynomials in $\operatorname{tr}\boldsymbol{e}^2$ and $\operatorname{tr}\boldsymbol{e}^3$ if φ_{ij} in (5.3) and (5.4) are polynomials in the arguments $\partial x_p/\partial X_q$.

6. Derivation from the Constitutive Equations for Materials with Memory

We consider a body to undergo the deformation described by (2.2). We assume that the material of which the body consists is such that the stress components t_{ij} at time t are single-valued functions of the displacement gradients at time t and continuous functionals of the displacement gradients in the time interval $\tau=0$ to $\tau=t$. Then, the stress components can be expressed with any desired accuracy by an expression of the form

$$\begin{aligned} t_{ij}= {} & f_{ij}^{(0)}(t)+\int_0^t f_{ijk_1l_1}^{(1)}(t,\tau_1)\,\frac{\partial u_{k_1}(\tau_1)}{\partial X_{l_1}}\,d\tau_1+ \\ & +\int_0^t\!\!\int_0^t f_{ijk_1l_1k_2l_2}^{(2)}(t,\tau_1,\tau_2)\,\frac{\partial u_{k_1}(\tau_1)}{\partial X_{l_1}}\,\frac{\partial u_{k_2}(\tau_2)}{\partial X_{l_2}}\,d\tau_1\,d\tau_2+ \\ & +\cdots, \end{aligned} \tag{6.1}$$

where $f_{ij}^{(0)}, f_{ijk_1l_1}^{(1)}, \ldots$, in addition to being continuous single-valued functions of the indicated variables are also single-valued functions of $\partial u_p(t)/\partial X_q$.

Now, if we consider only the class of deformations described by (2.3), with specified $f(\tau)$, we see that equation (6.1) becomes

$$\begin{aligned} t_{ij}= {} & f_{ij}^{(0)}(t)+\frac{\partial v_{k_1}}{\partial X_{l_1}}\int_0^t f_{ijk_1l_1}^{(1)}(t,\tau_1)\,f(\tau_1)\,d\tau_1+ \\ & +\frac{\partial v_{k_1}}{\partial X_{l_1}}\,\frac{\partial v_{k_2}}{\partial X_{l_2}}\int_0^t\!\!\int_0^t f_{ijk_1l_1k_2l_2}^{(2)}(t,\tau_1,\tau_2)\,f(\tau_1)\,f(\tau_2)\,d\tau_1\,d\tau_2+ \\ & +\cdots. \end{aligned} \tag{6.2}$$

Carrying out the integrations, we see that t_{ij} may be expressed as a single-valued function of $\partial v_p/\partial X_q$ and t, as indicated in (2.4).

From similar considerations it is evident that if we take as our starting point that t_{ij} are single-valued functions of the displacement gradients and their time derivatives of the first R orders (say) at time t and continuous functionals of these in the time interval $\tau=0$ to t, we arrive at a similar conclusion.

7. The Solution of a Simple Problem

If we have a body of the material considered and we know the constitutive equation (4.14) or (5.5), for a given function $f(t)$ in equation (2.3), then we can always calculate the forces necessary to produce in the body a specified deformation of the form (2.3), without further assumption regarding the precise manner in which the α's depend on the J's. In general, however, the calculated force system will involve body forces which could not, in practice, be readily applied, or will lack physical interest in some other respect.

However, in certain simple cases it can be shown that the deformation may be produced by surface tractions only and these may be calculated. As an example, we may consider the forces necessary to produce, in a rod of circular cross-section with radius a of incompressible material for which the constitutive equation for the class of deformations (2.3) has the form (5.5), a simultaneous small extension and torsion.

We take as our reference system a rectangular Cartesian coordinate system x, the x_3 axis of which coincides with the axis of the rod. If the fractional extension at time τ is $\varepsilon f(\tau)$ and the amount of torsion is $\psi f(\tau)$ and $\varepsilon f(\tau) \ll 1$ and $\psi x_3 f(\tau) \ll 1$, we may write the displacement components u_i in the rod at time t in the form (2.3) where v_i is given by

$$v_1 = -\psi X_2 X_3 - \tfrac{1}{2}\varepsilon X_1, \quad v_2 = \psi X_1 X_3 - \tfrac{1}{2}\varepsilon X_2, \quad v_3 = \varepsilon X_3. \tag{7.1}$$

We note that the relation (5.1), which expresses the fact that the volume of each element of the body is unchanged in the deformation, is satisfied. Substituting (7.1) in (3.13) and the equations so obtained in the constitutive equation (5.5), we see that the stress components at time t are given by

$$\begin{aligned} t_{11} &= -p - \tfrac{1}{2}\alpha_1\varepsilon + \tfrac{1}{4}\alpha_2(\varepsilon^2 + \psi^2 X_2^2), & t_{23} &= \tfrac{1}{2}\psi X_1(\alpha_1 + \tfrac{1}{2}\varepsilon\alpha_2), \\ t_{22} &= -p - \tfrac{1}{2}\alpha_1\varepsilon + \tfrac{1}{4}\alpha_2(\varepsilon^2 + \psi^2 X_1^2), & t_{31} &= -\tfrac{1}{2}\psi X_2(\alpha_1 + \tfrac{1}{2}\varepsilon\alpha_2), \\ t_{33} &= -p + \alpha_1\varepsilon + \alpha_2(\varepsilon^2 + \tfrac{1}{4}\psi^2 R^2) & t_{12} &= -\tfrac{1}{4}\psi^2 X_1 X_2 \alpha_2, \end{aligned} \tag{7.2}$$

where

$$R^2 = X_1^2 + X_2^2.$$

In these expressions, α_1 and α_2 are, of course, functions of t and of $\operatorname{tr}\boldsymbol{e}^2$ and $\operatorname{tr}\boldsymbol{e}^3$ given, from (7.1) and (3.13), by

$$\operatorname{tr}\boldsymbol{e}^2 = \tfrac{3}{2}\varepsilon^2 + \tfrac{1}{2}\psi^2 R^2 \quad \text{and} \quad \operatorname{tr}\boldsymbol{e}^3 = \tfrac{3}{4}\varepsilon^3 + \tfrac{3}{8}\varepsilon\psi^2 R^2. \tag{7.3}$$

If the function $f(\tau)$ varies sufficiently slowly with time, the deformation may be treated as a quasi-static deformation and inertial forces may be neglected in the equations of motion. With the approximation indicated by $\partial u_i/\partial X_j \ll 1$, we may then write these as

$$\frac{\partial t_{ij}}{\partial X_j} = 0. \tag{7.4}$$

With (7.2), and the assumption that the surface tractions are zero on the cylindrical surface of the rod, equation (7.4) yields

$$p = -(\tfrac{1}{2}\alpha_1\varepsilon - \tfrac{1}{4}\alpha_2\varepsilon^2) - \tfrac{1}{4}\psi^2 \int_a^R \alpha_2 R\, dR. \tag{7.5}$$

Introducing this expression for p into (7.2) we find that the surface tractions per unit area which must be applied to the ends of the rod in order to maintain the deformation have azimuthal components

$$\tfrac{1}{2}\psi R(\alpha_1 + \tfrac{1}{2}\varepsilon\alpha_2) \tag{7.6}$$

and longitudinal components

$$\tfrac{3}{2}\alpha_1\,\varepsilon + \tfrac{3}{4}\alpha_2\,\varepsilon^2 + \tfrac{1}{4}\alpha_2\,\psi^2 R^2 + \tfrac{1}{4}\psi^2 \int\limits_a^R \alpha_2 R\,dR. \tag{7.7}$$

The distribution of surface tractions (7.6) and (7.7) over an end of the rod is statically equivalent to a couple of magnitude

$$\pi\,\psi \int\limits_0^a R^3(\alpha_1 + \tfrac{1}{2}\,\varepsilon\,\alpha_2)\,dR \tag{7.8}$$

and a tensile force of magnitude

$$\pi \int\limits_0^a (3\,\alpha_1\,\varepsilon + \tfrac{3}{2}\alpha_2\,\varepsilon^2 + \tfrac{1}{4}\alpha_2\,\psi^2 R^2)\,R\,dR. \tag{7.9}$$

8. A Particular Form of the Constitutive Equation

Let us suppose that the form of the function $f(\tau)$ in (2.3) is specified, and that the constitutive equation (4.14) [or, in the particular case of an incompressible material, (5.5)] is valid for this form of $f(\tau)$. Now, let us specify a deformation of the type (2.3). In general, if a certain force system is appropriate for the maintenance of the deformation at one instant of time, a force system of quite a different type must be applied at other times. For example, if at one instant of time the deformation may be supported without application of body forces, then this will not generally be the case at other times.

However, if α_0, α_1 and α_2 have the forms

$$\alpha_0 = \varphi(t)\,\gamma_0, \quad \alpha_1 = \varphi(t)\,\gamma_1 \quad \text{and} \quad \alpha_2 = \varphi(t)\,\gamma_2, \tag{8.1}$$

where γ_0, γ_1 and γ_2 are independent of t, then, if the deformation can be maintained at one instant of time without application of body forces and with boundary conditions which take the form of specified displacement components or zero components of surface traction on portions of the surface which are specified, it can easily be seen that this will also be the case at all other times, provided tha that the deformation may be treated as quasi-static. For, the equations of motion and the boundary conditions may then be written in a form which is independent of t.

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On Circulation in Relativistic Hydrodynamics

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Abstract. In this paper it is shown that the relativistic analogue of the circulation in classical hydrodynamics is the integral

$$C=\int\left(1+\frac{i}{c^2}\right)U_\alpha\,d\,x^\alpha,$$

where i is the specific enthalpy of the fluid, U_α are the covariant components of the four velocity of the fluid and the integral is taken around a closed curve traveling with the fluid. This integral is similar to an integral given by LICHNEROWICZ, but differs in the factor multiplying U_α in the integrand.

The necessary and sufficient conditions for C to vanish are shown to be that the flow be irrotational and isentropic. The necessary and sufficient condition for C to be a constant of the motion is that the pressure be a function of the density alone at every point in space-time occupied by the fluid. This condition is shown to be violated when a shock is present. Results for similar integrals taken around vortex lines are also obtained. The relations between these results and Bernoulli's theorems are discussed.

1. Introduction

It is the purpose of this paper to discuss an integral formed from quantities describing the relativistic behavior of a perfect fluid. This integral is analogous to the integral defining circulation in classical hydrodynamics. The results obtained below are, in part, similar to results given by SYNGE [*1*] and reported by LICHNEROWICZ [*2*] but are obtained by using a different formulation for the stress-energy tensor of a perfect fluid. Our results hold for both special and general relativity and depend only on the assumed nature of the stress-energy tensor and the four conservation laws it obeys and the equation of conservation of matter.

A perfect fluid is described by the stress energy tensor

$$T^{\mu\nu}=\sigma\,U^\mu\,U^\nu-\frac{p}{c^2}\,g^{\mu\nu}, \tag{1.1}$$

where

$$U^\mu\,U_\mu=1; \tag{1.2}$$

U^μ represents the four dimensional velocity vector of the fluid, $g_{\mu\nu}$, the metric tensor of space-time, p, the pressure, and

$$\sigma\,c^2=\varrho\left(c^2+\varepsilon+\frac{p}{\varrho}\right). \tag{1.3}$$

In this equation ϱ is the proper density, and

$$\varepsilon = \varepsilon(p, \varrho) \tag{1.4}$$

is the proper, specific internal energy of the fluid. Equation (1.4) is sometimes referred to as the caloric equation of state of the fluid.

The equations governing the motion of the fluid are the formulation of five conservation laws, namely, the conservation of mass, energy and momentum. Where there are no singularities, these equations take the form of partial differential equations which are

$$\begin{aligned} T^{\mu\nu}{}_{;\nu} &= 0, \\ (\varrho U^{\mu})_{;\mu} &= 0, \end{aligned} \tag{1.5}$$

where the semicolon denotes the covariant derivative with respect to the metric tensor $g_{\mu\nu}$. This paper is not concerned with the determination of this tensor. It is assumed to be known.

Across a singular hyper-surface [3], the conservation laws are

$$\begin{aligned} [T^{\mu}{}_{\nu} \lambda_{\mu}] &= 0, \\ [\varrho U^{\mu} \lambda_{\mu}] &= 0, \end{aligned} \tag{1.6}$$

where λ_{μ} are the components of the unit normal vector to the singular hyper-surface, and we have used the notation

$$[f] = \lim_{\varepsilon \to 0} [f(x^{\mu} - \varepsilon \xi^{\mu}) - f(x^{\mu} + \varepsilon \xi^{\mu})] = f_{-} - f_{+},$$

where the x^{μ} are the coordinates of a point on the singular surface and the ξ^{μ} are arbitrary.

If equations (1.1) are substituted into the first four of equations (1.5), they may be written as (*cf.* [3])

$$U^{\mu}{}_{;\nu} U^{\nu} = \frac{p_{,\nu}}{\sigma c^2} (g^{\mu\nu} - U^{\mu} U^{\nu}) \tag{1.7}$$

and

$$T S_{;\mu} U^{\mu} = 0, \tag{1.8}$$

where T is the proper temperature, and S is the specific proper entropy. The quantities T and S are defined as in classical hydrodynamics by the requirement that

$$T\,dS = d\varepsilon + p\,d\left(\frac{1}{\varrho}\right). \tag{1.9}$$

We define the specific enthalpy as

$$i = \varepsilon + \frac{p}{\varrho}. \tag{1.10}$$

Then

$$di = T\,dS + \frac{dp}{\varrho}, \tag{1.11}$$

and

$$\sigma c^2 = \varrho (c^2 + i). \tag{1.12}$$

We further define

$$-\varphi = \int_{p_0}^{p} \frac{dp}{\sigma c^2},$$

where the integrand is considered a function of p and S, and the integration is carried out with S=constant. Then

$$-\varphi=\int\limits_{p_0}^{p}\frac{di}{c^2+i}=\log\frac{c^2+i(p,S)}{c^2+i(p_0,S)}. \tag{1.13}$$

We shall treat p_0 as independent of the coordinates of points of space-time in our subsequent discussion. Then

$$-\frac{\partial\varphi}{\partial x^\mu}=+\frac{1}{\sigma c^2}\frac{\partial p}{\partial x^\mu}+\frac{\partial\varphi}{\partial S}\frac{\partial S}{\partial x^\mu}, \tag{1.14}$$

where p is kept constant in the partial differentiation of φ with respect to S.

For some functions $\varepsilon(p,\varrho)$ there exists a p_0 such that

$$i(p_0,S)=0.$$

With this value of p_0, equation (1.13) becomes

$$e^{-\varphi}=1+\frac{i}{c^2} \tag{1.15}$$

and

$$\frac{\partial e^{-\varphi}}{\partial S}=-e^{-\varphi}\frac{\partial\varphi}{\partial S}=\frac{T}{c^2}.$$

2. The Tensor $\Omega_{\mu\nu}$

We define the antisymmetric tensor

$$\Omega_{\mu\nu}=V_{\mu;\nu}-V_{\nu;\mu}, \tag{2.1}$$

where

$$V_\mu=e^{-\varphi}U_\mu, \tag{2.2}$$

and φ is given by equation (1.13). This tensor will play an important role in our subsequent discussion. We, therefore, discuss some of its properties. It follows from equation (2.1) and (2.2) that

$$\Omega_{\mu\nu}=e^{-\varphi}[W_{\mu\nu}-(\varphi_{,\nu}U_\mu-\varphi_{,\mu}U_\nu)],$$

where

$$W_{\mu\nu}=U_{\mu;\nu}-U_{\nu;\mu}, \tag{2.3}$$

and hence

$$W_{\mu\nu}U^\nu=U_{\mu;\nu}U^\nu \tag{2.4}$$

in view of equation (1.2).

We further define the vorticity vector v^μ by the equation

$$v^\mu=\tfrac{1}{2}E^{\mu\nu\sigma\tau}W_{\sigma\tau}U_\nu, \tag{2.5}$$

where

$$E^{\mu\nu\sigma\tau}=\frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\sigma\tau} \tag{2.6}$$

and

$$E_{\mu\nu\sigma\tau}=\sqrt{-g}\,\varepsilon_{\mu\nu\sigma\tau}. \tag{2.7}$$

The relative tensors $\varepsilon_{\mu\nu\sigma\tau}=\varepsilon^{\mu\nu\sigma\tau}$ are zero unless all indices are different. In that case, they are plus or minus one, depending on whether the values taken by the indices are an even or an odd permutation of 1 2 3 4.

It is a consequence of the definition of these quantities that

$$E_{\mu_1\dots\mu_k\nu_1\dots\nu_r}\,E^{\mu_1\dots\mu_k\lambda_1\dots\lambda_r} = k!\,\delta^{\lambda_1\dots\lambda_r}_{\nu_1\dots\nu_r}, \tag{2.8}$$

where the tensor $\delta^{\lambda_1\dots\lambda_r}_{\nu_1\dots\nu_r}$ is zero unless the indices $\lambda_1\dots\lambda_r$ are an even or an odd permutation of the indices $\nu_1\dots\nu_r$. In the former case it has the value $+1$, and in the latter case, the value -1.

Multiplying equation (2.5) by $E_{\mu\alpha\beta\gamma}$ and summing, we obtain

$$v^\mu E_{\mu\alpha\beta\gamma} = \tfrac{1}{2}\,\delta^{\nu\sigma\tau}_{\alpha\beta\gamma}\,W_{\sigma\tau}\,U_\nu = U_\alpha W_{\beta\gamma} + U_\beta W_{\gamma\alpha} + U_\gamma W_{\alpha\beta}. \tag{2.9}$$

It then follows that

$$W_{\beta\gamma} = v^\mu U^\alpha E_{\mu\alpha\beta\gamma} - U_\beta U_{\gamma;\alpha} U^\alpha + U_\gamma U_{\beta;\alpha} U^\alpha. \tag{2.10}$$

Substituting equation (1.7) into equation (2.10) and the resulting equation into the expression for $\Omega_{\mu\nu}$ in terms of $W_{\mu\nu}$, we obtain

$$\Omega_{\beta\gamma} = e^{-\varphi}\left[v^\mu U^\alpha E_{\mu\alpha\beta\gamma} + \frac{\partial\varphi}{\partial S}\,(S_{;\gamma}U_\beta - S_{;\beta}U_\gamma)\right] \tag{2.11}$$

after having made use of equation (1.14). When equation (1.15) holds, this may be written as

$$\Omega_{\beta\gamma} = v^\mu V^\alpha E_{\mu\alpha\beta\gamma} - \frac{T}{c^2}\,(S_{;\gamma}U_\beta - S_{;\beta}U_\gamma). \tag{2.12}$$

It is immediately evident that if

$$v^\mu = 0, \quad \text{and} \quad S = \text{constant}, \tag{2.13}$$

then $\Omega_{\beta\gamma}=0$. That is, if the flow is irrotational and isentropic, then $\Omega_{\beta\gamma}=0$. The converse also follows; for, if the right-hand side of equation (2.11) vanishes, we may multiply by U^γ and sum to obtain

$$\left(\frac{\partial e^{-\varphi}}{\partial S}\right) S_{;\beta} = 0$$

in view of equation (1.8). Hence,

$$v^\mu U^\alpha E_{\mu\alpha\beta\gamma} = 0.$$

This implies that

$$v^\mu = A\,U^\mu.$$

However, it is a consequence of the definition of v^μ (*cf.* equation (2.5)) that

$$v^\mu U_\mu = 0. \tag{2.14}$$

Hence, the first of equations (2.13) also holds. Thus, equations (2.13) are the necessary and sufficient conditions for

$$\Omega_{\mu\nu} = 0.$$

3. Algebraic Properties of $\boldsymbol{\Omega_{\mu\nu}}$

Associated with the tensor $\Omega_{\mu\nu}$ there is the dual tensor defined by the equation

$$\breve{\Omega}_{\mu\nu} = \tfrac{1}{2}\,E^{\mu\nu\lambda\varrho}\,\Omega_{\lambda\varrho}. \tag{3.1}$$

It is a consequence of equations (2.8) and (2.12) that

$$\breve{\Omega}^{\mu\nu} = -\left(V^\mu v^\nu - v^\mu V^\nu + \frac{T}{c^2}\,S_{,\varrho}\,U_\lambda\,E^{\mu\nu\lambda\varrho}\right). \tag{3.2}$$

Hence
$$\Omega_1 = \Omega_{\mu\nu} \breve{\Omega}^{\mu\nu} = 4 \frac{T}{c^2} e^{-\varphi} v^\mu S_{,\mu}. \tag{3.3}$$

Thus, the tensor $\Omega_{\mu\nu}$ will be singular if, and only if, $v^\mu T S_{,\mu} = 0$. If it is singular, it can only be of rank zero or two.

Another invariant associated with the tensor $\Omega_{\mu\nu}$ is
$$\Omega_2 = \Omega_{\mu\nu} \Omega^{\mu\nu} = 2 \left[e^{-2\varphi} v^\mu v_\mu + \frac{T^2}{c^4} S_{,\mu} S_{,\nu} g^{\mu\nu} \right]. \tag{3.4}$$

Since both v^μ and $T S_{,\mu}$ are space-like vectors (*cf.* equations (2.15) and (1.8)), the quantity Ω_2 satisfies
$$\Omega_2 \leqq 0. \tag{3.5}$$

The equality holds if, and only if,
$$v_\mu = 0, \quad \text{and} \quad S_{,\mu} = 0,$$
that is, if, and only if,
$$\Omega_{\mu\nu} = 0.$$

It is a consequence of equations (2.11), (1.2), and (1.8) that
$$\begin{aligned} \Omega_{\mu\nu} U^\nu &= \frac{T}{c^2} S_{,\mu}, \\ \Omega_{\mu\nu} v^\nu &= -\frac{T}{c^2} S_{,\nu} v^\nu U_\mu, \\ \Omega_{\mu\nu} \frac{T}{c^2} S_{,\alpha} g^{\alpha\nu} &= w_\mu - \left(\frac{T^2}{c^4} g^{\alpha\beta} S_{,\alpha} S_{,\beta} \right) U_\mu, \end{aligned} \tag{3.6}$$
where
$$w_\mu = E_{\mu\nu\sigma\tau} v^\nu V^\sigma \frac{T}{c^2} g^{\tau\alpha} S_{,\alpha} \tag{3.7}$$
and is a vector orthogonal to u^μ, v^μ and $T S_{,\mu}$. It further follows that
$$\Omega_{\mu\nu} w^\nu = e^{-2\varphi} \left[\left(v^\sigma \frac{T}{c^2} S_{,\sigma} \right) v_\mu - (v_\alpha v^\alpha) \frac{T}{c^2} S_{,\mu} \right] \tag{3.8}$$
and that
$$w_\nu w^\nu = e^{-2\varphi} \left[(v^\alpha v_\alpha) \left(\frac{T^2}{c^4} S_{,\beta} g^{\beta\gamma} S_{,\gamma} \right) - \left(\frac{T}{c^2} v^\gamma S_{,\gamma} \right)^2 \right]. \tag{3.9}$$

In case $\Omega_1 = 0$, that is, $\Omega_{\mu\nu}$ is singular but not all of its components vanish (v^μ and $T S_{,\mu}$ are not both zero), it follows from the above that
$$\Omega_{\mu\nu} v^\nu = 0$$
and
$$\Omega_{\mu\nu} y^\nu = 0,$$
where
$$y^\mu = \left(U^\mu + \frac{e^{2\varphi}}{v^\alpha v_\alpha} w^\mu \right). \tag{3.10}$$

That is, the vectors v^ν and y^ν are annulled by the matrix of coefficients of the tensor $\Omega_{\mu\nu}$. The vector y^μ reduces to U^μ whenever w^μ vanishes, that is, whenever
$$v^\alpha = A \frac{T S_{,\gamma}}{c^2} g^{\gamma\alpha},$$
where A is an arbitrary scalar.

Since

$$y^\mu y_\mu = 1 + \frac{e^{4\varphi}}{(v^\alpha v_\alpha)^2} w_\mu w^\mu$$

and $w_\mu w^\mu \geqq 0$ as follows from equation (3.9) and the application of SCHWARZ's inequality to the space-like vectors involved, the vector y^μ is always time-like.

4. Tubes in Space-Time

If $z^\mu(x)$ is a vector field, the solutions of the ordinary differential equations

$$\frac{d x^\mu}{dW} = z^\mu(x) \tag{4.1}$$

define a three parameter family of curves passing through a hypersurface Σ in space-time. We shall assume that the hypersurface may be represented by the equation

$$x^{i_4} = f(x^{i_1}, x^{i_2}, x^{i_3}) \tag{4.2}$$

where i_1, i_2, i_3 and i_4 are some permutation of the numbers 1 2 3 4.

We may then write the solutions of equations (4.1) as

$$x^\mu = x^\mu(\xi^i, W) \qquad (i = i_1, i_2, i_3), \tag{4.3}$$

where the ξ^i are the initial values of the x^i, that is

$$x^i(\xi^i, 0) = \xi^i$$

and

$$x^{i_4}(\xi^i, 0) = f[x^{i_1}(\xi^i, 0), x^{i_2}(\xi^i, 0), x^{i_3}(\xi^i, 0)].$$

If in the surface Σ we have a closed curve described by the equations

$$\xi^i = \xi^i(\tau), \qquad \tau_1 \leqq \tau \leqq \tau_2$$

with

$$\xi^i(\tau_1) = \xi^i(\tau_2),$$

then the equations

$$x^\mu = x^\mu[\xi^i(\tau), W] = x^\mu(\tau, W) \tag{4.4}$$

define a two-dimensional surface in space-time which we shall call a tube. The closed curves obtained by setting constant values for W into equation (4.4) are images of the closed curve on the initial surface. When in equations (4.4) τ is constant, the equations define a curve which is a solution of equation (4.1).

The tangent vector to the curves of parameter τ is given by

$$\lambda^\mu = \frac{\partial x^\mu}{\partial \tau} \tag{4.5}$$

where W is kept fixed in the differentiation. We have

$$\frac{\partial \lambda^\mu}{\partial W} = \frac{\partial^2 x^\mu}{\partial W\, \partial \tau} = \frac{\partial z^\mu}{\partial \tau}. \tag{4.6}$$

When the vector field $z^\mu(x)$ is set equal to the four-velocity $U^\mu(x)$, W is the proper-time and the curves of parameter W are called the world-lines of particles making up the fluid. The four variables ξ^i, W may be interpreted as a new set of coordinates in space-time, co-moving coordinates. They are the analogues of Lagrange coordinates in classical hydrodynamics. The curves of parameter τ (W fixed) are then said to "travel with the fluid".

Another example of a tube in space-time is obtained by setting $z^\mu(x)$ equal to the vorticity vector $v^\mu(x)$. Then the curves of parameter W given by equation (4.3) are said to be vortex-lines, and the tube is called a vortex tube.

5. Invariant Integrals

Consider the integral

$$\Gamma(W) = \int_{\tau_1}^{\tau_2} V_\alpha[x(\tau, W)]\,\lambda^\alpha\,d\tau, \tag{5.1}$$

where V_α is given by equation (2.2) and is considered as a function of τ and W in view of equation (4.4) and λ^α is the tangent vector to a curve of parameter W given by equation (4.5). The integral $\Gamma(W)$ represents the integral of the vector V_α which is proportional to the velocity vector, around a closed curve on the tube defined by the solutions of equation (4.1).

We now compute

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} \left(\frac{\partial V_\alpha}{\partial x^\sigma}\, z^\sigma \lambda^\alpha + V_\alpha \frac{\partial \lambda^\alpha}{\partial W} \right) d\tau .$$

This may be written as

$$\begin{aligned} \frac{d\Gamma}{dW} &= \int_{\tau_1}^{\tau_2} \left(\Omega_{\alpha\sigma}\, z^\sigma \lambda^\alpha + \frac{\partial V_\sigma}{\partial x^\alpha}\, \lambda^\alpha z^\sigma + V_\alpha \frac{\partial \lambda^\alpha}{\partial W} \right) d\tau \\ &= \int_{\tau_1}^{\tau_2} \left(\Omega_{\alpha\sigma}\, z^\sigma \lambda^\alpha + \frac{\partial V_\sigma}{\partial \tau}\, z^\sigma + V_\alpha \frac{\partial z^\alpha}{\partial \tau} \right) d\tau \end{aligned}$$

where we have made use of equation (4.6). Thus

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} \left[\Omega_{\alpha\sigma}\, z^\sigma \lambda^\alpha + \frac{\partial}{\partial \tau} (V_\sigma z^\sigma) \right] d\tau .$$

Since we are integrating around a closed curve, we finally obtain

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} \Omega_{\alpha\sigma}\, z^\sigma \lambda^\alpha\, d\tau . \tag{5.2}$$

The necessary and sufficient condition for $\Gamma(W) = 0$ for tubes defined by the vector field z^μ by means of arbitrary initial surfaces and arbitrary curves in this surface is that

$$\Omega_{\alpha\beta} = 0 . \tag{5.3}$$

Similarly, the necessary and sufficient condition that $d\Gamma/dW = 0$ for tubes defined with the same amount of generality is that

$$Z_{\alpha,\beta} - Z_{\beta,\alpha} = 0$$

where

$$Z_\alpha = \Omega_{\alpha\sigma}\, z^\sigma . \tag{5.4}$$

6. The Circulation

In this section we set $z^\alpha = U^\alpha$, the four-velocity vector and $W = s$, the proper time. We shall then name the integral in equation (5.1) the circulation and denote it by $C(s)$. Equation (5.2) then becomes

$$\frac{dC}{ds} = -\int_{\tau_1}^{\tau_2} e^{-\varphi} \frac{\partial \varphi}{\partial S} S_{,\alpha} \lambda^\alpha \, d\tau \tag{6.1}$$

in view of the first of equations (3.6).

Thus, a necessary and sufficient condition for the circulation to be independent of proper-time is that

$$\left(e^{-\varphi} \frac{\partial \varphi}{\partial S} S_{;\alpha}\right)_{;\beta} = \left(e^{-\varphi} \frac{\partial \varphi}{\partial S} S_{;\beta}\right)_{;\alpha}. \tag{6.2}$$

It is clear that for isentropic flows ($S = \text{constant}$), $d\Gamma/ds = 0$.

When equation (1.15) holds, equation (6.2) may be written as

$$(T S_{;\alpha})_{;\beta} - (T S_{;\beta})_{;\alpha} = 0. \tag{6.3}$$

In virtue of equation (1.9) this may be written as

$$p_{;\alpha}\, \varrho_{;\beta} - p_{;\beta}\, \varrho_{;\alpha} = 0. \tag{6.4}$$

Hence, in this case the necessary and sufficient condition that $dC/ds = 0$ is that at every point of space-time

$$p = p(\varrho). \tag{6.5}$$

That is, the motion is such that a knowledge of p (or ϱ) at a point determines ϱ (or p).

7. Bernoulli's Theorem (Time-dependent Flows)

The definition of the circulation, namely,

$$C(S) = \int_{\tau_1}^{\tau_2} e^{-\varphi} U_\alpha \lambda^\alpha \, d\tau$$

involves an integrand which, aside from the exponential factor, is the tangential component of the four-velocity field U^α taken at a point of a curve which travels with the fluid. $C(s)$ is, thus, similar to the classical circulation. To show that it is a generalization of this classical quantity, we shall show that as in classical hydrodynamics, the condition

$$C(s) = 0 \tag{7.1}$$

and is related to Bernoulli's theorem for time-dependent irrotational and isentropic flows.

It follows from equation (5.3) that the condition (7.1) is equivalent to

$$\Omega_{\mu\nu} = 0,$$

that is, to the requirement that the flow be isentropic and irrotational (*cf.* Section 2). Under these conditions (*cf.* [3]) there exists a scalar function $R(x)$ such that

$$V_\mu = e^{-\varphi} U_\mu = \frac{\partial R}{\partial x^\mu}. \tag{7.2}$$

Define the quantity

$$\frac{q^2}{c^2} = -g_{\mu\nu} U^\mu U^\nu + \frac{U_4^2}{g_{44}} = -\left(g_{\mu\nu} - \frac{g_{4\mu} g_{4\nu}}{g_{44}}\right) U^\mu U^\nu.$$

This is the length of the vector 0^μ which is given by

$$0^\mu = U^\mu - \frac{\delta_4^\mu U_4}{g_{44}}$$

and which is the projection of U^μ onto the three-space orthogonal to the curves of parameter x^4.

In view of equation (1.2) we have

$$U_4 = \sqrt{g_{44}} \sqrt{1 + \frac{q^2}{c^2}}. \tag{7.3}$$

Equations (7.2) with $\mu = 4$ imply that

$$e^{-\varphi} U_4 = e^{-\varphi} \sqrt{g_{44}} \sqrt{1 + \frac{q^2}{c^2}} = \frac{\partial R}{\partial x^4}.$$

When equation (1.15) holds, this becomes

$$\left(1 + \frac{i}{c^2}\right) \sqrt{g_{44}} \sqrt{1 + \frac{q^2}{c^2}} = \frac{\partial R}{\partial x^4}.$$

If we now write

$$g_{44} = 1 + \frac{2V}{c^2} \tag{7.4}$$

and expand the left-hand side of the above equation neglecting powers of $1/c$ higher than the second, we obtain

$$\left(i + V + \frac{1}{2} q^2\right) = c^2 \left(\frac{\partial R}{\partial x^4} - 1\right). \tag{7.5}$$

This is the classical form of the Bernoulli equation for a non-steady irrotational, isentropic flow in the gravitational field with potential V. The quantity $c^2 R$ is to be interpreted in terms of the velocity potential for the flow.

Equation (7.4) is the usual relation between the classical Newtonian potential and the metric tensor of space-time which holds in the case of weak fields.

In case we use equation (1.13), equation (7.5) becomes

$$\left(i + V + \frac{1}{2} q^2\right) = [c^2 + i(p_0, S)] \left(\frac{\partial R}{\partial x^4} - 1\right). \tag{7.6}$$

8. Bernoulli's Theorem (Steady Flows)

In classical theory Bernoulli's theorem is usually formulated for stationary flows. LICHNEROWICZ [2] has indicated a method of describing steady flows of fluid in their own gravitational fields. It consists of assuming that the space-time and the velocity field are invariant under a time-like one-parameter group of motions. That is, there exists a vector field ξ^α which is such that Killing's equations

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \tag{8.1}$$

are satisfied and such that

$$\begin{aligned} U^\tau \xi^\sigma{}_{;\tau} - \xi^\tau U^\sigma{}_{;\tau} &= 0, \\ S_{;\alpha} \xi^\alpha = p_{;\alpha} \xi^\alpha &= 0. \end{aligned} \tag{8.2}$$

Equations (8.2) state that the velocity field U^σ is invariant under the group of motions as are the entropy and pressure.

Define the quantity

$$H = e^{-\varphi} U_\alpha \xi^\alpha = V_\alpha \xi^\alpha \tag{8.3}$$

and compute

$$H_{;\beta} U^\beta = (e^{-\varphi} U_\alpha \xi^\alpha)_{;\beta} U^\beta .$$

We have

$$\begin{aligned} H_{;\beta} U^\beta &= e^{-\varphi}(-\varphi_{;\beta} U^\beta U_\alpha \xi^\alpha + U_{\alpha;\beta} U^\beta \xi^\alpha + U_\alpha \xi^\alpha_{;\beta} U^\beta) \\ &= e^{-\varphi}\left[\frac{p_{,\alpha}}{\sigma c^2}\xi^\alpha + \frac{1}{2} U^\alpha U^\beta (\xi_{\alpha;\beta} + \xi_{\beta;\alpha})\right] = 0 \end{aligned}$$

as a consequence of equations (1.7), (1.8), (1.14), (8.1) and (8.2). Thus, along a world-line of the fluid we have

$$H = \text{constant}.$$

If the coordinate system is chosen so that

$$\xi^\alpha = \delta^\alpha_4$$

this equation becomes

$$e^{-\varphi} U_4 = \text{constant}.$$

Using the results of the preceding section, we may write this as

$$i + \frac{q^2}{2} + V = \text{constant} \tag{8.4}$$

along a world-line of the fluid. It may very from world-line to world-line. Equation (8.4) is the usual form of the Bernoulli equation.

It is instructive to compute the derivative of H in an arbitrary direction. Thus,

$$\begin{aligned} H_{;\beta} &= e^{-\varphi}[-\varphi_{,\beta} U_\alpha \xi^\alpha + U_{\alpha;\beta}\xi^\alpha + U_\alpha \xi^\alpha_{;\beta}] \\ &= e^{-\varphi}[e^\varphi \Omega_{\alpha\beta}\xi^\alpha - \varphi_{,\alpha}\xi^\alpha U_\beta + U_{\beta;\alpha}\xi^\alpha - U^\alpha \xi_{;\alpha} + U^\alpha(\xi_{\alpha;\beta} + \xi_{\beta;\alpha})] \end{aligned}$$

as follows from the definition of $\Omega_{\alpha\beta}$ (*cf.* equation (3.2)). In view of equations (8.2) we have

$$H_{;\beta} = \Omega_{\alpha\beta}\xi^\alpha .$$

Hence, if $\Omega_{\alpha\beta} = 0$,

$$H = \text{constant}$$

thoughout space-time. That is, if the motion is isentropic and irrotational, the Bernoulli constant is independent of the world-line of the fluid particle as is the case in classical theory.

9. Vortex Tubes

In this section we shall discuss the integral (5.1) when the tubes in question are vortex tubes, that is, when we set $z^\mu(x) = v^\mu(x)$, and the motion is rotational ($v^\mu \neq 0$). In this case, we cannot have $\Gamma(W) = 0$. However, it follows from equations (5.2) and (3.6) that

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} e^{-\varphi}\frac{\partial\varphi}{\partial S} S_{,\nu} v^\nu U_\alpha \lambda^\alpha \, d\tau . \tag{9.1}$$

Equation (9.1) may be written as

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} h V_\alpha \lambda^\alpha \, d\tau \tag{9.2}$$

with

$$h = \frac{\partial \varphi}{\partial S} S_{,\nu} v^\nu . \tag{9.3}$$

The necessary and sufficient condition for $d\Gamma/dW = 0$ for tubes defined by arbitrary curves in these surfaces is, then,

$$(h V_\alpha)_{;\beta} - (h V_\beta)_{;\alpha} = 0 .$$

This may be written as

$$h \Omega_{\alpha\beta} + h_{,\beta} V_\alpha - h_{,\alpha} V_\beta = 0 .$$

Thus,

$$h \Omega_{\alpha\beta} = h_{,\alpha} V_\beta - h_{,\beta} V_\alpha$$

and is of rank 2. Therefore, $h = 0$ or $\Omega_1 = 0$. However, $\Omega_1 = e^\varphi h$. Hence, the necessary and sufficient condition for $d\Gamma/dW = 0$ for vortex tubes defined as above is that $\Omega_1 = 0$, and $\Omega_{\alpha\beta}$ be of rank 2.

In this case we also have

$$\frac{d\Gamma}{dW} = 0$$

for world tubes defined by the vector field y^μ given by equation (3.10).

10. The Change in Circulation Across Shocks

All of the preceding discussion was based on the assumption that the derivatives of the flow variables existed everywhere. We now consider the possibility of singular hyper-surfaces in space-time, representing shocks across which equations (1.6) obtain. These equations may be written as

$$\frac{m}{c} = \varrho_+ U_+^\mu \lambda_\mu = \varrho_- U_-^\mu \lambda_\mu \tag{10.1}$$

and

$$\frac{m}{c} (\mu_+ U_+^\mu - \mu_- U_-^\mu) = \frac{\lambda^\mu}{c^2} (p_+ - p_-) , \tag{10.2}$$

where

$$\mu = \frac{\sigma}{\varrho} = 1 + \frac{i}{c^2} . \tag{10.3}$$

In case equation (1.15) holds, equation (10.2) may be written as

$$\frac{m}{c} (V_+^\mu - V_-^\mu) = \frac{\lambda^\mu}{c^2} (p_+ - p_-) , \tag{10.4}$$

where λ^μ is normalized so that

$$\lambda^\mu \lambda_\mu = -1 \tag{10.5}$$

and for a shock wave $m \neq 0$.

It then follows from these equations that the components of the vector field V_μ tangent to the hyper-surface are continuous across it. As has been shown earlier, the components normal to the hyper-surface are discontinuous. We may determine the discontinuity in this component in terms of the discontinuity in pressure (or density) across the shock as follows: multiply equations (10.2) by $U_{+\mu}$, $U_{-\mu}$ and λ_μ to obtain

$$(\mu_+ - \mu_- U_-^\mu U_{+\mu}) = \frac{1}{c^2}\,\frac{1}{\varrho_+}\,(p_+ - p_-)\,, \tag{10.6}$$

$$(\mu_+ U_+^\mu U_{-\mu} - \mu_-) = \frac{1}{c^2}\,\frac{1}{\varrho_-}\,(p_+ - p_-)\,, \tag{10.7}$$

$$\frac{m^2}{c^2}\left(\frac{\mu_+}{\varrho_+} - \frac{\mu_-}{\varrho_-}\right) = -\frac{1}{c^2}\,(p_+ - p_-)\,, \tag{10.8}$$

respectively. If we now eliminate $U_-^\mu U_{+\mu}$ from the first two of these equations we have the equation

$$\mu_+^2 - \mu_-^2 = \frac{1}{c^2}\,(p_+ - p_-)\left(\frac{\mu_+}{\varrho_+} + \frac{\mu_-}{\varrho_-}\right). \tag{10.9}$$

From equations (10.7) and (10.1) we have

$$\frac{m}{c} = \frac{1}{c}\left(\frac{p_+ - p_-}{\dfrac{\mu_-}{\varrho_-} - \dfrac{\mu_+}{\varrho_+}}\right)^{\frac{1}{2}} = \frac{\varrho_+}{\mu_+}\,(V_+^\mu \lambda_\mu) = \frac{\varrho_-}{\mu_-}\,(V_-^\mu \lambda_\mu)\,. \tag{10.10}$$

Equations (10.9) and (10.10) are the relativistic Rankine-Hugoniot equations previously derived [4]. When μ is known as a function of p and ϱ, equation (10.9) may be used to determine p_+ in terms of p_-, ϱ_+ and ϱ_-. Equations (10.10) may then be used to express ϱ_+ and hence p_+ as a function of $V_-^\mu \lambda_\mu$, p_- and ϱ_-. They may also be used to express $V_+^\mu \lambda_\mu$ as a function of the latter quantities.

We shall write

$$M = \frac{\mu_+}{\mu_-}\,, \qquad \eta = \frac{\varrho_+}{\varrho_-}\,, \quad \text{and} \quad y = \frac{p_+}{p_-}\,. \tag{10.11}$$

Then equations (10.9) and (10.10) may be written as

$$\begin{aligned} M^2 - 1 &= \left(1 - \frac{M^2}{\eta^2}\right)(V_-^\mu \lambda_\mu) = \alpha^2 (y - 1)\left(\frac{M}{\eta} + 1\right), \\ V_+^\mu \lambda_\mu &= \frac{M}{\eta}\,(V_-^\mu \lambda_\mu)\,, \\ y - 1 &= \left(1 - \frac{M}{\eta}\right)\left[\frac{(V_-^\mu \lambda_\mu)^2}{\alpha^2}\right], \\ \alpha^2 &= \frac{p_-}{c^2\,\varrho_-\,\mu_-}\,. \end{aligned} \tag{10.12}$$

When μ is known as a function of p and ϱ, the first of these equations may be used to determine y in terms of η. We may then use the remaining equations to determine all quantities as functions of $V_-^\mu \lambda_\mu$. Since the entropy will not be conserved across shocks, care must be taken to choose solutions of these equations such that

$$S_+ > S_-$$

across the singular hyper-surface.

When a shock is such that λ_μ is a function of the points of space-time, we say that it is of varying strength. This variation may be due to the curvature of the shock at each time $x^4 =$ constant or to the dependence of λ_μ on x^4 alone. The world-lines of particles which cut the singular hyper-surface have discontinuous tangents at the hyper-surface. Along such a world-line the entropy suffers a discontinuity. The size of this discontinuity in entropy varies from world-line to world-line. As a result, it is not true that the pressure and density at all points of space-time are related as in equation (6.5). Thus, when shocks are present, we cannot have $dC/ds=0$ for arbitrary tubes.

This result may be seen in another manner. Suppose we have a shock of varying strength (λ^μ not constant) and suppose it is such that V_-^μ, p_-, and ϱ_- are constant. Then, we would have

$$\Omega^-_{\mu\nu} = V_{-\mu;\nu} - V_{-\nu;\mu} = 0.$$

However if

$$\Omega^+_{\mu\nu} = 0,$$

then it follows from results of Section 2 that the flow behind the shock would be both isentropic and irrotational. However, we have seen that it is not isentropic. Thus,

$$\Omega^+_{\mu\nu} \neq 0.$$

This means that we can construct tubes of world-lines which cross shocks and for which $C(S)=0$ on one portion of the tube, but $C(S)\neq 0$ on other portions.

The detailed behavior of the circulation across shocks depends on the explicit expression obtained when the system of equations (10.12) are solved as described above. These expressions depend in turn on the nature of μ as a function of p and ϱ or equivalently on the nature of ε as a function of p and ϱ. Once this function is prescribed, we may study the geometry of the world-lines in terms of the geometry of the shock surface by introducing a coordinate system consisting of the world-lines of the fluid particles as curves of parameter x^4 and the singular hyper-surface as the hyper-surface $x^4=$constant. The methods used in studying stationary and pseudo-stationary flows [5] may then be applied in a straightforward manner.

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On the Convergence of the Rayleigh Quotient Iteration for the Computation of the Characteristic Roots and Vectors. III

(Generalized Rayleigh Quotient and Characteristic Roots with Linear Elementary Divisors)

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In this part[*] of our paper we give a generalization of the Rayleigh quotient iteration method which can be applied in most cases to non-symmetric matrices (in the complex case, to non-Hermitian matrices). The enumeration of the sections and formulae continues that of Parts I and II.

25. The reason why the Rayleigh quotient method works out so well in the symmetric case is to be found in its connection with certain variational problems, since in the symmetric case the eigenvalues are the values of the extrema of the usual Rayleigh quotient, corresponding to the eigenvectors.

In the case of non-symmetric matrices, however, the quadratic form in question has to be replaced by a bilinear form. But in this case no extremal property analogous to the former one is known. However, as a matter of fact, the reason for the good convergence of the Rayleigh quotient iteration lies not in the extremal property itself but in the *stationary property* of the Rayleigh quotient in the neighborhood of an eigenvector. Under these circumstances it can be asked whether the stationary property still holds in the case of a generalized Rayleigh quotient, and this turns out indeed to be the fact under fairly general conditions. In the corresponding cases the generalized Rayleigh quotient iteration remains convergent also for non-symmetric matrices; the convergence turns out to be cubic, and even the main result (46) of the symmetric case remains generally true in the non-symmetric case.

26. For a general square matrix, A, of order n, we shall call the expression

$$R(\xi,\eta) = \frac{\eta A \xi'}{\eta \xi'} \tag{72}$$

the *generalized Rayleigh quotient* (to be denoted by G. R. Q.) corresponding to the vectors ξ and η.

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We assume through the end of Section 28 that A is real. Corresponding to a real eigenvalue σ of A, let ξ_0 and η_0 be (real) right and left eigenvectors of A:

$$A\,\xi_0' = \sigma\,\xi_0', \qquad \eta_0 A = \sigma\,\eta_0. \tag{73}$$

Assume further that

$$\eta\,\xi' \neq 0. \tag{74}$$

Then we verify at once that

$$R(\xi_0, \eta_0) = \sigma. \tag{75}$$

We ask now whether $R(\xi, \eta)$ has at the point (ξ_0, η_0) the stationary property with respect to (ξ, η), that is to say, whether the partial derivatives of $R(\xi, \eta)$ with respect to the components of ξ and η vanish at the point (ξ_0, η_0). We shall show that this is indeed the case if σ is a single or multiple eigenvalue of A, *provided that all corresponding elementary divisors are linear.* On the other hand, we shall show by an example that if a non-linear elementary divisor corresponds to σ, then the stationary property need not be satisfied.

27. Assume that σ is an eigenvalue of A of multiplicity m with only linear elementary divisors. Then we can introduce special coordinates in the ξ-space and in the η-space in such a way that the first m coordinate vectors in the ξ-space become a complete set of m linearly independent eigenvectors of A from the right and that the same property holds also for the η-space. Then, if the components of ξ and η are correspondingly denoted by $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, we have

$$\eta\,A\,\xi' = \sigma(x_1 y_1 + \cdots + x_m y_m) + B(x_{m+1}, \ldots, x_n;\ y_{m+1}, \ldots, y_n),$$

where $B(x_{m+1}, \ldots, x_n; y_{m+1}, \ldots, y_n)$ is a bilinear form in the variables x_ν and y_ν written out as arguments.

Now let ξ and η tend to the eigenvectors ξ_0, η_0, where

$$\xi_0 = (a_1, \ldots, a_m;\ 0 \ldots 0), \qquad \eta_0 = (b_1, \ldots, b_m;\ 0 \ldots 0);$$
$$\xi = (a_1, \ldots, a_m;\ \alpha_{m+1}, \ldots, \alpha_n), \qquad \eta = (b_1, \ldots, b_m;\ \beta_{m+1}, \ldots, \beta_m)$$

and $\eta_0\,\xi_0' = a_1 b_1 + \cdots + a_m b_m \neq 0$. Suppose further that we have for $\varepsilon \to 0$, $\alpha_\mu = O(\varepsilon)$, $\beta_\mu = O(\varepsilon)$ $(\mu = m+1, \ldots, n)$. Then we have

$$\eta\,A\,\xi' = \sigma(a_1 b_1 + \cdots + a_m b_m) + O(\varepsilon^2),$$
$$\eta\,\xi' = a_1 b_1 + \cdots + a_m b_m + O(\varepsilon^2),$$
$$R(\xi, \eta) - \sigma = \frac{O(\varepsilon^2)}{\eta\,\xi'},$$

and since

$$\eta\,\xi' \to \eta_0\,\xi_0' \neq 0,$$

we have finally

$$R(\xi, \eta) = \sigma + O(\varepsilon^2). \tag{76}$$

This result is the indication of the stationary property in this case. To give the formal proof, observe that we have

$$R(\xi, \eta) - \sigma = \frac{B(\alpha_{m+1}, \ldots, \alpha_n; \beta_{m+1}, \ldots, \beta_n) - \sigma(\alpha_{m+1}\beta_{m+1} + \cdots + \alpha_n\beta_n)}{(a_1 b_1 + \cdots + a_m b_m) + (\alpha_{m+1}\beta_{m+1} + \cdots + \alpha_n\beta_n)};$$

but here the numerator on the right is of *second order* with respect to the coordinates, which vanish in (ξ_0, η_0), while the denominator tends to $\eta_0 \xi_0'$. This proves our first assertion.

28. To prove the second one, consider the matrix

$$A = \begin{pmatrix} \sigma & 1 \\ 0 & \sigma \end{pmatrix}$$

which has σ as a double eigenvalue and the elementary divisor $(\lambda - \sigma)^2$. The eigenvectors are from the right $(1, 0)$ and from the left $(0, 1)$. Here, if $\xi = (x_1, x_2)$, $\eta = (y_1, y_2)$, the G.R.Q. is

$$\frac{\sigma(x_1 y_1 + x_2 y_2) + x_1 y_2}{x_1 y_1 + x_2 y_2}.$$

For $\xi = (1, \alpha)$, $\eta = (\beta, 1)$ with α, β tending to 0, this becomes

$$\sigma + \frac{1}{\alpha + \beta},$$

and there is no trace of the stationary property at the point $\xi_0 = (1, 0)$, $\eta_0 = (0, 1)$.

29. The generalized Rayleigh quotient iteration corresponding to G.R.Q. defined by (72) is given by the formulae

$$\lambda_k = R(\xi_k, \eta_k), \tag{77}$$

$$\begin{aligned} (A - \lambda_k I)\, \xi_{k+1}' &= \xi_k', \\ \eta_{k+1}(A - \lambda_k I) &= \eta_k, \end{aligned} \tag{78}$$

which are to be applied consecutively for $k = 0, 1, \ldots$, starting with a couple ξ_0, η_0. These starting vectors are chosen arbitrarily, subject to the condition $\eta_0 \xi_0' \neq 0$*.

Let σ be an eigenvalue of A to which correspond only *linear* elementary divisors. Then we are going to prove:

Theorem 1. *There exist a positive $\delta^{(0)} = \delta(\xi_0, \eta_0, A)$ depending on ξ_0, η_0, and A, and a certain algebraic function of the components of the vectors ξ and η, $p_1(\xi, \eta)$, depending only on A and the choice of σ, such that if we have*

$$|\lambda_0 - \sigma| \leqq \delta^{(0)}, \tag{79}$$

$$p_1(\xi_0, \eta_0) \neq 0, \tag{80}$$

then the sequence λ_k tends to σ.

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* Another approach to the generalized Rayleigh quotient iteration has been given by M. R. Hestenes, Inversion of matrices by biorthogonalization and related results, J. Soc. Indust. Appl. Math. **6**, 51—90 (1958), who derived it by applying Newton's method of differential corrections simultaneously to eigenvalues and to the corresponding couples of eigenvectors. From this point of view the cubic convergence of our method is particularly interesting, since Newton's method usually gives only *quadratic convergence* [see, *e.g.*, A. Ostrowski, Konvergenzdiskussion und Fehlerabschätzung für die Newtonsche Methode bei Gleichungssystemen. Comm. Math. Helv. **9**, 79—103 (1936 37) and A. Ostrowski, Simultaneous systems of equations. National Bureau of Standards, Appl. Math. Ser. **29**, 29—34 (1953)].

Theorem 2. *If the sequence λ_k tends to σ, without any of the λ_k becoming $=\sigma$, and ξ_0, η_0 satisfy the condition* (80), *then we have*

$$\lambda_{k+1}-\sigma = o[(\lambda_k-\sigma)^2] \qquad (k\to\infty); \tag{81}$$

moreover, for any fixed integer p,

$$\lambda_{k+1}-\sigma = o[(\lambda_k-\sigma)^2(\lambda_{k-1}-\sigma)^2\dots(\lambda_{k-p}-\sigma)^2] \qquad (k\to\infty). \tag{82}$$

Theorem 3. *Suppose that under the hypotheses of Theorem 2, one of the two following conditions is satisfied:*

A. *Denote by σ_ν all eigenvalues of A different from σ and different from each other. Consider among these σ_ν those whose distance from σ is minimal and denote them by $\sigma_1, \sigma_2, \dots, \sigma_t$. Let L be the greatest exponent of an elementary divisor corresponding to $\sigma_1, \sigma_2, \dots, \sigma_t$, and assume that to only one among $\sigma_1, \sigma_2, \dots, \sigma_t$ really corresponds an elementary divisor with exponent L*[*]. *Further, assume that a certain algebraic function of the components of ξ_0 and η_0, $p_3(\xi_0, \eta_0)$, depending only on A, ξ_0, η_0 and σ, does not vanish.*

B. *All eigenvalues of A are real, and A has only linear elementary divisors.*

Then we have

$$\frac{\lambda_{k+1}-\sigma}{(\lambda_k-\sigma)^3} \to g \neq 0 \qquad (k\to\infty). \tag{83}$$

30. Assume that σ is a characteristic root of exact multiplicity $m \geqq 1$ and denote the elementary divisors of the matrix A corresponding to the eigenvalues $\neq \sigma$, in a convenient order, by

$$(\lambda-\sigma_1)^{n_1}, \dots, (\lambda-\sigma_l)^{n_l}, \tag{84}$$

$$m + \sum_{\nu=1}^{l} n_\nu = n, \qquad \operatorname*{Max}_{\nu} n_\nu = N. \tag{85}$$

If A is transformed by a matrix S to the Jordan canonical form, we obtain for $S^{-1}A\,S$ the representation

$$\sigma I_m \dotplus \sum_{\nu=1}^{l}{}^{\boldsymbol\cdot}\, T_{n_\nu}, \tag{86}$$

$$T_{n_\nu} = \sigma_\nu I_{n_\nu} + U_{n_\nu}, \tag{87}$$

where generally I_k denotes the unity matrix of order k and U_k the so-called *auxiliary unity matrix of order k* consisting of units in the first superdiagonal and zeros elsewhere. The symbols $\dotplus$, $\sum^{\boldsymbol\cdot}$ denote the direct sum of matrices aligned along the main diagonal. In this transformation the vectors η_k are to be replaced by $\eta_k S^{-1}$, and the vectors ξ'_k by $S\xi'_k$. The inner products $\eta_k\xi'_k$ are then, of course, invariant, and the same holds for the generalized Rayleigh quotient defined by (72), and for the equations (78), and of course also for the values of λ_k given by (77).

We can therefore assume from the beginning that A has the form (86).

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[*] Of course this does not exclude the possibility that this elementary divisor is multiple.

31. Let the decomposition of the vectors ξ_k and η_k corresponding to the decomposition (86) of the matrix A be

$$\xi_k = \xi_1^{(k)} \dotplus \sum_{\nu=1}^{l}\!\cdot\ \xi_{2,\nu}^{(k)}, \qquad \eta_k = \eta_1^{(k)} \dotplus \sum_{\nu=1}^{l}\!\cdot\ \eta_{2,\nu}^{(k)} \tag{88}$$

and put

$$p_1 = p_1^{(0)} = p_1(\xi_0, \eta_0) = \tfrac{1}{2}\,\eta_1^{(0)}\,\xi_1^{(0)\prime}, \tag{89}$$

$$\begin{gathered} B = \sum_{\nu=1}^{l}\!\cdot\ T_{n_\nu}, \\ \xi_2^{(k)} = \sum_{\nu=1}^{l}\!\cdot\ \xi_{2,\nu}^{(k)}, \qquad \eta_2^{(k)} = \sum_{\nu=1}\!\cdot\ \eta_{2,\nu}^{(k)}. \end{gathered} \tag{90}$$

Then we have from (78)

$$\xi_1^{(k+1)} = \frac{1}{\sigma-\lambda_k}\,\xi_1^{(k)}, \qquad \eta_1^{(k+1)} = \frac{1}{\sigma-\lambda_k}\,\eta_1^{(k)} \qquad (k = 0, 1, \ldots), \tag{91}$$

$$(B-\lambda_k I)\,\xi_2'^{(k+1)} = \xi_2'^{(k)}, \qquad \eta_2^{(k+1)}(B-\lambda_k I) = \eta_2^{(k)} \qquad (k = 0, 1, \ldots) \tag{92}$$

From (91), putting

$$M_k = \prod_{\varkappa=0}^{k}(\sigma-\lambda_\varkappa) \qquad (k = 0, 1, \ldots), \tag{93}$$

we have

$$\xi_1^{(k)} = \frac{1}{M_{k-1}}\,\xi_1^{(0)}, \qquad \eta_1^{(k)} = \frac{1}{M_{k-1}}\,\eta_1^{(0)}. \tag{94}$$

32. From the decomposition (86) of A and (94), (89), we have

$$\eta_k A\,\xi_k' = \frac{2p_1\sigma}{M_{k-1}^2} + Z^{(k)}, \qquad Z^{(k)} = \eta_2^{(k)}\,B\,\xi_2^{(k)\prime}, \tag{95}$$

$$\eta_k\,\xi_k' = \frac{2p_1}{M_{k-1}^2} + N^{(k)}, \qquad N^{(k)} = \eta_2^{(k)}\,\xi_2^{(k)\prime} \tag{96}$$

and from (77), (72)

$$\lambda_k - \sigma = \frac{Z^{(k)} - \sigma N^{(k)}}{\dfrac{2p_1}{M_{k-1}^2} + N^{(k)}} = M_{k-1}^2 D_k, \tag{97}$$

$$D_k = \frac{Z^{(k)} - \sigma N^{(k)}}{2p_1 + M_{k-1}^2 N^{(k)}}. \tag{98}$$

Now put

$$N_\nu^{(k)} = \eta_{2,\nu}^{(k)}\,\xi_{2,\nu}^{(k)\prime}, \qquad Z_\nu^{(k)} = \eta_{2,\nu}^{(k)}\,T_{n_\nu}\,\xi_{2,\nu}^{(k)\prime}. \tag{99}$$

Then from (90) and (92) we have, since U_m commutes with E_m,

$$(T_{n_\nu} - \lambda_k I_{n_\nu})\,\xi_{2,\nu}^{(k+1)\prime} = \xi_{2,\nu}^{(k)\prime}, \qquad \eta_{2,\nu}^{(k+1)}(T_{n_\nu} - \lambda_k I_{n_\nu}) = \eta_{2,\nu}^{(k)},$$

$$\xi_{2,\nu}^{(k)\prime} = \prod_{\varkappa=0}^{k-1}(T_{n_\nu} - \lambda_\varkappa I_{n_\nu})^{-1}\,\xi_{2,\nu}^{(0)\prime}, \qquad \eta_{2,\nu}^{(k)} = \eta_{2,\nu}^{(0)}\prod_{\varkappa=0}^{k-1}(T_{n_\nu} - \lambda_\varkappa I_{n_\nu})^{-1},$$

$$N_\nu^{(k)} = \eta_{2,\nu}^{(0)}\prod_{\varkappa=0}^{k-1}(T_{n_\nu} - \lambda_\varkappa I_{n_\nu})^{-2}\,\xi_{2,\nu}^{(0)\prime}, \tag{100}$$

$$Z_\nu^{(k)} = \eta_{2,\nu}^{(0)}\,T_{n_\nu}\prod_{\varkappa=0}^{k-1}(T_{n_\nu} - \lambda_\varkappa I_{n_\nu})^{-2}\,\xi_{2,\nu}^{(0)\prime}. \tag{101}$$

From (90) and the definitions of $Z^{(k)}$ and $N^{(k)}$ we have now by (99)

$$Z^{(k)}=\sum_{\nu=1}^{l} Z_\nu^{(k)}, \qquad N^{(k)}=\sum_{\nu=1}^{l} N_\nu^{(k)}, \qquad Z^{(k)}-\sigma N^{(k)}=\sum_{\nu=1}^{l}\left(Z_\nu^{(k)}-\sigma N_\nu^{(k)}\right). \tag{102}$$

Further, from (97) and the definition of M_k in (93)

$$\frac{\lambda_{k+1}-\sigma}{(\lambda_k-\sigma)^3}=\frac{D_{k+1}}{D_k}. \tag{103}$$

33. In what follows we shall have to use the notion of the *bound of a matrix*. The bound $\Lambda(C)$ of an $n\times n$ matrix C is defined by

$$\Lambda(C)=\max|C\,\xi| \qquad (|\xi|=1). \tag{104}$$

From this definition, we see at once that if the matrix C is *majorized by a matrix* C_1 with non-negative elements, $C\ll C_1$, that is to say, if the modulus of each element of C does not exceed the corresponding element of C_1, we have

$$\Lambda(C)\leqq\Lambda(C_1) \qquad (C\ll C_1). \tag{105}$$

From (104) follows the *triangular inequality*,

$$\Lambda(C_1+C_2)\leqq\Lambda(C_1)+\Lambda(C_2) \tag{106}$$

and the *homogeneity* for any scalar, c,

$$\Lambda(c\,C)=|c|\,\Lambda(C), \tag{107}$$

as well as

$$\Lambda(C_1C_2)\leqq\Lambda(C_1)\,\Lambda(C_2). \tag{108}$$

It is well known and easily proved that $\Lambda(C)^2$ is the greatest eigenvalue of the matrix $C\,C^*$. From this it is seen at once that $\Lambda(E)=1$ and it can be easily deduced for the auxiliary matrix U, by forming $U\,U'$, that $\Lambda(U)=1$[*]. We obtain now, for any scalar, w,

$$\Lambda(w\,I+U)\leqq\Lambda(w\,I)+\Lambda(U)\leqq|w|+1. \tag{109}$$

Further, by using (105), (106), (107), for any scalar w, with $|w|\geqq 1$, since $U^n=0$,

$$\Lambda\left(\frac{1}{w\,I+U}\right)=\frac{1}{|w|}\,\Lambda\left(I-\frac{U}{w}+\frac{U^2}{w^2}-\cdots\pm\frac{U^{n-1}}{w^{n-1}}\right)\leqq\sum_{\nu=0}^{n-1}\Lambda(U^\nu)=n,$$

$$\Lambda\left(\frac{1}{w\,I_n+U_n}\right)\leqq n \qquad (|w|\geqq 1). \tag{110}$$

34. Now put

$$\operatorname*{Min}_{\sigma_\nu\neq\sigma}|\sigma_\nu-\sigma|=2d. \tag{111}$$

The number d is obviously a homogeneous function of the elements of A in the sense that if all elements of A are multiplied by the scalar c, then d is multiplied by $|c|$, while σ and σ_ν are multiplied by c, and the same is true of the λ_k deduced from (77) and (72). The three assertions stated in Section 29 remain invariant if all elements of A are multiplied by $c \neq 0$, though of course the values of the constants $\delta^{(0)}$ and g occurring in these assertions are changed. We can therefore without loss of generality assume that

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[*] *Cf.* OSTROWSKI, A. M.: On the bounds of a one-parametric family of matrices. Journal f. d. r. u. a. Mathematik **200**, 190—199 (1958).

$$|\sigma_\nu - \sigma| \geqq 2 \qquad (\sigma_\nu \neq \sigma). \tag{112}$$

Further, put

$$p_2^{(0)} = p_2(\xi_0, \eta_0) = \sum_{\nu=1}^{l} |\eta_{2,\nu}^{(0)} \xi_{2,\nu}^{(0)}| = \sum_{\nu=1}^{l} |N_\nu^{(0)}|, \tag{113}$$

$$p = p(\xi_0, \eta_0) = \frac{|p_1^{(0)}|}{p_2^{(0)}}. \tag{114}$$

Now we make the assumption that the vectors ξ_0, η_0, though arbitrarily chosen, satisfy the inequality

$$p_1^{(0)} \neq 0. \tag{115}$$

If one of the vectors ξ_0, η_0 is already an eigenvector corresponding to σ, then we have $p_2 = 0$, $p = \infty$. Therefore the size of p is to a certain degree an indication of the nearness of one of the vectors ξ_0, η_0 to a corresponding eigenvector. On the other hand, if one of the vectors ξ_0, η_0 had only vanishing components corresponding to the eigenvalue σ, we should have $p = 0$, and this is excluded by (115).

35. Now put

$$\operatorname*{Max}_{\sigma_\nu \neq \sigma} |\sigma_\nu - \sigma| = d_1 - 1. \tag{116}$$

From (100) and (101) we have

$$Z_\nu^{(k)} - \sigma N_\nu^{(k)} = \eta_{2,\nu}^{(0)} (T_{n_\nu} - \sigma I_{n_\nu}) \cdot \prod_{\varkappa=0}^{k-1} (T_{n_\nu} - \lambda_\varkappa I_{n_\nu})^{-2} \xi_{2,\nu}^{(0)}. \tag{117}$$

Here, by (109) and (87), we have obviously

$$\Lambda(T_{n_\nu} - \sigma I_{n_\nu}) \leqq |\sigma_\nu - \sigma| + 1 \leqq d_1. \tag{118}$$

We now make the assumption that for a fixed value k considered in this discussion we have

$$|\lambda_\varkappa - \sigma| \leqq 1 \qquad (\varkappa = 0, 1, \ldots, k-1). \tag{119}$$

Then from (112) follows

$$|\sigma_\nu - \lambda_\varkappa| \geqq 1 \qquad (\varkappa = 0, 1, \ldots, k-1;\ \nu = 1, 2, \ldots, l). \tag{120}$$

Using this we obtain from (110) and (85)

$$\Lambda\left(\frac{1}{T_{n_\nu} - \lambda_\varkappa I_{n_\nu}}\right) = \Lambda\left(\frac{1}{(\sigma_\nu - \lambda_\varkappa) I_{n_\nu} + U_{n_\nu}}\right) \leqq N \quad (\varkappa = 0, 1, \ldots, k-1;\ \nu = 1, \ldots, l). \tag{121}$$

Using (108), we now obtain

$$\Lambda\left[(T_{n_\nu} - \sigma I_{n_\nu}) \prod_{\varkappa=0}^{k-1} (T_{n_\nu} - \lambda_\varkappa I_{n_\nu})^{-2}\right] \leqq d_1 N^{2k}. \tag{122}$$

From the definition of Λ and (117) we have

$$|Z_\nu^{(k)} - \sigma N_\nu^{(k)}| \leqq d_1 n_\nu^{2k} |\eta_{2,\nu}^{(0)}| \, |\xi_{2,\nu}^{(0)}| \tag{123}$$

and, further, from (102) and (113),

$$|Z^{(k)} - \sigma N^{(k)}| \leqq d_1 \sum_{\nu=1}^{l} n_\nu^{2k} |\eta_{2,\nu}^{(0)}| \, |\xi_{2,\nu}^{(0)}| < d_1 p_2^{(0)} N^{2k}. \tag{124}$$

Moreover, again using (121) and (110) we have

$$\Lambda\left[\prod_{\varkappa=1}^{k} (T_{n_\nu} - \lambda_\varkappa I_{n_\nu})^{-1}\right] \leqq n_\nu^{2k}, \tag{125}$$

and therefore by (100), (102) and (113)

$$|N_\nu^{(k)}| \leqq n_\nu^{2k} |\eta_{2,\nu}^{(0)} \xi'^{(0)}_{2,\nu}| = n_\nu^{2k} |N_\nu^{(0)}|, \tag{126}$$

$$|N^{(k)}| \leqq \sum_{\nu=1}^{l} n_\nu^{2k} |N_\nu^{(0)}| \leqq p_2^{(0)} N^{2k}. \tag{127}$$

36. We have now from (98) by (124) and (127)

$$|D_k| < \frac{d_1 p_2^{(0)} N^{2k}}{2|p_1^{(0)}| - p_2^{(0)} N^{2k} |M_{k-1}|^2}. \tag{128}$$

Define δ by the formula

$$\delta = \delta(\xi_0, \eta_0) = \frac{\operatorname{Min}\left(\frac{1}{2}, \frac{p}{d_1}\right)}{N^2} \tag{129}$$

and assume, instead of (119), that the sharper relation

$$|\sigma - \lambda_\varkappa| \leqq \delta \qquad (\varkappa = 0, \ldots, k-1) \tag{130}$$

is verified.

Then by (93)

$$|M_{k-1}| \leqq \delta^k,$$

and we obtain from (128) by (114)

$$|D_k| \leqq \frac{d_1 p_2^{(0)} N^{2k}}{2|p_1^{(0)}| - p_2^{(0)} (\delta N)^{2k}} = \frac{d_1 N^{2k}}{2p - \delta N^2 \cdot \delta \cdot (\delta N)^{2k-2}} \leqq \frac{d_1 N^{2k}}{2p - \delta N^2}.$$

It follows, since $d_1 \geqq 1$ and therefore by (129) $\delta N^2 \leqq p$, that

$$|D_k| < \frac{d_1}{p} N^{2k}.$$

We see that by (97)

$$|\lambda_k - \sigma| \leqq \frac{d_1}{p} N^{2k} |M_{k-1}|^2, \tag{131}$$

and therefore

$$\begin{aligned} |\lambda_k - \sigma| &\leqq \frac{d_1}{p} (\delta N)^{2k} \\ &= \delta \left(\frac{d_1 \delta N^2}{p}\right) (\delta N^2)^{k-1} \delta^{k-1}. \end{aligned}$$

But here in the right-hand expression by (129) the second and third factors are both $\leqq 1$, and therefore we obtain

$$|\lambda_k - \sigma| \leqq \delta^k \leqq \delta. \tag{132}$$

From (130) it follows, therefore, that the same inequality is also satisfied for λ_k. Therefore (132) remains true if we assume instead of (130) only

$$|\sigma - \lambda_0| \leqq \delta. \tag{133}$$

37. In order to improve (132), we note that by (132)

$$|M_{k-1}|^2 \leqq \delta^2 \prod_{\varkappa=1}^{k-1} \delta^{2\varkappa} = \delta^{k^2-k+2} \tag{134}$$

and therefore from (131)

$$|\lambda_k - \sigma| \leqq \left(\frac{d_1}{p}\,\delta N^2\right)(\delta N^2)^{k-1}\,\delta^{(k-1)^2+1};$$

here, by (129), the factors in parentheses on the right are $\leqq 1$. Thus we obtain

$$|\lambda_k - \sigma| \leqq \delta^{(k-1)^2+1} \qquad (k = 1, 2, \ldots). \tag{135}$$

Putting $k+1$ instead of k in (131) and dividing both sides by the product

$$(\lambda_k - \sigma)^2\,(\lambda_{k-1} - \sigma)^2 \ldots (\lambda_{k-t} - \sigma)^2,$$

for a fixed $t < k$ and by the definition of M_k we obtain obviously

$$\left|\frac{\lambda_{k+1} - \sigma}{\prod\limits_{\tau=0}^{t} (\lambda_{k-\tau} - \sigma)^2}\right| \leqq \frac{d_1}{p} N^{2k+2}\,|M_{k-t-1}|^2. \tag{136}$$

The expression on the right, by the definition of M_k and by (135), is

$$\leqq \frac{d_1}{p} N^{2k+2}\,|\lambda_{k-t-1} - \sigma|^2 \leqq \frac{d_1}{p} N^{2k+2}\,\delta^{2(k-t-2)^2+2} = \frac{d_1}{p}\,(\delta N^2)^{k+1}\,\delta^{2k^2-(4t+9)k+1+2(t+2)^2},$$

and by (129) this is

$$\leqq \delta^{2k^2-(4t+9)k+2(t+2)^2+1}$$

and tends to zero with $k \to \infty$.

The convergence condition (133) with the expression (129) for δ has been derived under the assumption that d as given by (111) has the value 1. If we now drop this assumption, then the general case $d \gtreqless 1$ is reduced to the case $d = 1$ by replacing λ_k, σ, σ_ν and d_1 by $\frac{\lambda_k}{d}$, $\frac{\sigma}{d}$, $\frac{\sigma_\nu}{d}$, $\frac{d_1+d-1}{d}$. The condition (133) then becomes

$$\left|\frac{\sigma}{d} - \frac{\lambda_0}{d}\right| \leqq \frac{1}{N^2}\,\mathrm{Min}\left(\frac{1}{2}, \frac{p\,d}{d_1+d-1}\right),$$

or

$$|\sigma - \lambda_0| \leqq \frac{d}{N^2}\,\mathrm{Min}\left(\frac{1}{2}, \frac{p\,d}{d_1+d-1}\right),$$

and we see that in the general case the number $\delta^{(0)}$ used in (79) is defined by

$$\delta^{(0)} = \frac{d}{N^2}\,\mathrm{Min}\left(\frac{1}{2}, \frac{p\,d}{d_1+d-1}\right) = \delta(\xi_0, \eta_0, \sigma, A). \tag{137}$$

The initial vectors ξ_0, η_0 enter into the bound $\delta^{(0)}$ given by (137) only in the expression p defined by (113) and (114).

Our Theorem 1 of Section 29 is now completely proved.

It may be observed finally that if the number N defined by (85) is not known, it can be replaced by n, the order of A. On the other hand, if A has only linear elementary divisors, we have $N=1$.

The iteration defined by (77) and (78) starts with a couple of vectors ξ_0, η_0, and condition (79) implies that ξ_0 and η_0 are a more or less good approximation to certain eigenvectors of A corresponding to σ. Indeed, in this case the formulation of the stationary property discussed in Section 27 shows that λ_0 will be then a good approximation to σ and condition (79) is likely to be satisfied.

If, however, we wish to start with an approximation to σ, we denote it by λ_{-1} and from (78) applied to $k=-1$, derive the values of ξ_0 and η_0, taking for ξ_{-1} and η_{-1} arbitrary vectors. Then, as we have already discussed in a previous publication, the vectors ξ_0 and η_0 can be expected to be good approximations to certain eigenvectors of A and our discussion will apply.

38. We are now going to prove Theorem 2. In fact, the assertions (81) and (82) of this theorem have already been proved by formula (136), but only under the conditions of Theorem 1 and not under those of Theorem 2. It is therefore sufficient, in order to prove Theorem 2, to show that if $\lambda_k \to \sigma$ and (80) is satisfied, then for a certain k we have

$$|\lambda_k - \sigma| \leqq \delta^{(k)} \equiv \delta(\xi_k, \eta_k, A), \tag{138}$$

and

$$p_1(\xi_k, \eta_k) \neq 0. \tag{139}$$

But we have from (89) and (94)

$$p_1^{(k)} \equiv p_1(\xi_k, \eta_k) = \frac{1}{2}\,\eta_1^{(k)}\,\xi_1^{(k)\prime} = \frac{p_1(\xi_0, \eta_0)}{M_{k-1}^2}, \tag{140}$$

and since by (80) the numerator does not vanish, (139) is certainly satisfied.

In order to discuss $\delta(\xi_k, \eta_k)$, observe that by (113) and (126) we have

$$p_2^{(k)} \equiv p_2(\xi_k, \eta_k) = \sum_{\nu=1}^{l} |N_\nu^{(k)}| \leqq n^{2k} \sum_{\nu=1}^{l} |N_\nu^{(0)}| = p_2^{(0)} n^{2k}, \tag{141}$$

and by (114) and (140)

$$p^{(k)} \equiv p(\xi_k, \eta_k) = \frac{|p_1^{(k)}|}{p_2^{(k)}} \geqq \frac{1}{n^{2k} M_{k-1}^2}\,\frac{|p_1^{(0)}|}{p_2^{(0)}}. \tag{142}$$

Here, since $\lambda_k \to \sigma$ $(k \to \infty)$, we have $n^{2k} M_{k-1}^2 \to 0$; therefore

$$\delta^{(k)} \equiv \delta(\xi_k, \eta_k) = \frac{d}{N^2}\,\mathrm{Min}\left(\frac{1}{2}, \frac{p^{(k)} d}{d_1 + d_{-1}}\right)$$

from a certain k on becomes equal to $d/(2N^2)$, and therefore for a certain $k=k_0$ we surely have

$$|\sigma - \lambda_{k_0}| < \delta^{(k_0)}. \tag{143}$$

But then if we put $\lambda_k' = \lambda_{k+k_0}$, the assumptions under which Theorem 2 has been proved are satisfied for λ_k', and the assertions of Theorem 2, being true for λ_k', then follow immediately also for the sequence λ_k. Theorem 2 has now been proved*.

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* An analogous argument has to be added in Section 17 of Part I of this series, where the corresponding part of the proof was not developed in all necessary detail.

39. In proving Theorem 3 of Section 29 under the hypothesis that $\lambda_k \to \sigma$, we can again assume, without loss of generality, that (79) is true.

Before discussing the limit in (83), we prove the following

Lemma. *If I and U are the unity matrix and the auxiliary unity matrix of order n, and if a_k is an infinite sequence tending to $a \neq 0$ as $k \to \infty$, then*

$$\frac{1}{k^{n-1}} \prod_{\varkappa=1}^{k} \frac{1}{I - a_\varkappa U} \to \frac{a^{n-1}}{(n-1)!} U^{n-1} \qquad (k \to \infty). \tag{144}$$

Proof. For sufficiently small positive x, put

$$\prod_{\varkappa=1}^{k} \frac{1}{1 - a_\varkappa x} = \sum_{\mu=0}^{\infty} S_\mu^{(k)} x^\mu, \tag{145}$$

where $S_\mu^{(k)} = S_\mu^{(k)}(a_1, \ldots, a_k)$ is a symmetric polynomial in $a_1, \ldots, a_k$, homogeneous of degree μ. Then obviously, since I and U commute and $U^n = 0$

$$\prod_{\varkappa=1}^{k} \frac{1}{I - a_\varkappa U} = \sum_{\mu=0}^{n-1} S_\mu^{(k)} U^\mu,$$

and (144) follows at once if we prove that as $k \to \infty$,

$$\frac{S_\mu^{(k)}}{k^\mu} \to \frac{a^\mu}{\mu!} \qquad (\mu = 0, 1, \ldots, n-1). \tag{146}$$

Put

$$\frac{a_\varkappa}{a} = b_\varkappa \to 1 \qquad (\varkappa = 1, 2, \ldots).$$

Then (146) is equivalent to

$$\frac{S_\mu^{(k)}(b_1, \ldots, b_k)}{k^\mu} \to \frac{1}{\mu!} \qquad (k \to \infty). \tag{147}$$

Therefore from the beginning we can assume $a = 1$, $a_\varkappa \to 1$ and of course $\mu > 0$.

40. Put
$$s_\nu^{(k)} = a_1^\nu + \cdots + a_k^\nu = (1 + \varepsilon_{\nu,k})\, k \qquad (\nu = 0, 1, \ldots),$$

where $\varepsilon_{\nu,k} \to 0$ $(k \to \infty)$. Then we have

$$\prod_{\varkappa=1}^{k} \frac{1}{1 - a_\varkappa x} = \exp \sum_{\nu=1}^{\infty} \frac{s_\nu^{(k)} x^\nu}{\nu},$$

and therefore

$$\exp\left[k \sum_{\nu=1}^{n} (1 + \varepsilon_{\nu,k}) \frac{x^\nu}{\nu}\right] = 1 + \sum_{\nu=1}^{\infty} A_\nu^{(k)} x^\nu, \tag{148}$$

where

$$A_\nu^{(k)} = S_\nu^{(k)} \qquad (\nu = 1, 2, \ldots, n). \tag{149}$$

Putting here $x = y/k$ we obtain further

$$\exp\left\{(1 + \varepsilon_{1,k})\, x + \sum_{\nu=2}^{n} (1 + \varepsilon_{\nu,k}) \frac{y^\nu}{\nu k^{\nu-1}}\right\} = 1 + \sum_{\nu=1}^{\infty} \frac{A_\nu^{(k)}}{k^\nu} y^\nu. \tag{150}$$

As $k \to \infty$, the expression on the left-hand side tends to e^x, and the coefficients of its development in powers of x tend to the correspondent coefficients of e^x. We obtain $A_\nu^{(k)}/k^\nu \to 1/\nu!$, and (147) follows from (149).

41. From (117) and (87) we have

$$Z_\nu^{(k)} - \sigma N_\nu^{(k)} = \eta_{2,\nu}^{(0)} \left((\sigma_\nu - \sigma) I_{n_\nu} + U_{n_\nu} \right) \prod_{\varkappa=0}^{k-1} \left((\sigma_\nu - \lambda_\varkappa) I_n + U_{n_\nu} \right)^{-2} \xi_{2,\nu}'^{(0)},$$

$$Z_\nu^{(k)} - \sigma N_\nu^{(k)} = \eta_{2,\nu}^{(0)} \left((\sigma_\nu - \sigma) I_{n_\nu} + U_{n_\nu} \right) \prod_{\varkappa=0}^{k-1} \frac{1}{(\sigma_\nu - \lambda_\varkappa)^2} \Pi_k \, \xi_{2,\nu}'^{(0)}, \tag{151}$$

where

$$\Pi_k = \prod_{\varkappa=0}^{k-1} \left(I_{n_\nu} + \frac{1}{\sigma_\nu - \lambda_\varkappa} U_{n_\nu} \right)^{-2}. \tag{152}$$

But here we have

$$a_{2\varkappa+1} = a_{2\varkappa+2} \frac{-1}{\sigma_\nu - \lambda_\varkappa} \to \frac{1}{\sigma - \sigma_\nu} = a,$$

and therefore our lemma can be applied to the product Π_k. Observing that the number of factors in Π_k is $2k$, from our lemma we have

$$\frac{\Pi_k}{(2k)^{n_\nu - 1}} \to \frac{(\sigma - \sigma_\nu)^{-n_\nu+1}}{(n_\nu - 1)!} U_{n_\nu}^{n_\nu - 1} \qquad (k \to \infty).$$

Since $U_{n_\nu}^{n_\nu} = 0$ as $k \to \infty$, we have from (151)

$$\frac{Z_\nu^{(k)} - \sigma N_\nu^{(k)}}{k^{n_\nu - 1}} \prod_{\varkappa=0}^{k-1} (\sigma_\nu - \lambda_\varkappa)^2 \to \frac{(-2)^{n_\nu - 1} (\sigma_\nu - \sigma)^{-n_\nu+1}}{(n_\nu - 1)!} \left(\eta_{2,\nu}^{(0)} U_{n_\nu}^{n_\nu - 1} \xi_{2,\nu}'^{(0)} \right). \tag{153}$$

For the product on the left-hand side of (153) we have

$$\prod_{\varkappa=0}^{k-1} (\sigma_\nu - \lambda_\varkappa) = (\sigma_\nu - \sigma)^k \prod_{\varkappa=0}^{k-1} \left(1 + \frac{\sigma - \lambda_\varkappa}{\sigma_\nu - \sigma} \right) = (\sigma_\nu - \sigma)^k P_{\nu,k},$$

where

$$P_{\nu,k} = \prod_{\varkappa=0}^{k-1} \left(1 + \frac{\sigma - \lambda_\varkappa}{\sigma_\nu - \sigma} \right) \to P_\nu \equiv \prod_{\varkappa=0}^{\infty} \left(1 + \frac{\sigma - \lambda_\varkappa}{\sigma_\nu - \sigma} \right) \qquad (k \to \infty). \tag{154}$$

Therefore from (153) we have

$$\frac{Z_\nu^{(k)} - \sigma N_\nu^{(k)}}{k^{n_\nu - 1}} (\sigma_\nu - \sigma)^{2k + n_\nu - 1} \to \frac{(-2)^{n_\nu - 1}}{[(n_\nu - 1)!]^2 P_\nu^2} \left(\eta_{2,\nu}^{(0)} U_{n_\nu}^{n_\nu - 1} \xi_{2,\nu}'^{(0)} \right) \qquad (k \to \infty). \tag{155}$$

42. We assume now that the conditions under A of Theorem 3 in Section 29 are satisfied. Then, if the σ_ν in (84) are suitably ordered, we have

$$\operatorname*{Min}_\nu |\sigma_\nu - \sigma| = |\sigma_1 - \sigma| = \cdots = |\sigma_s - \sigma| < |\sigma_{s+1} - \sigma| \leqq \cdots \leqq |\sigma_l - \sigma|, \tag{156}$$

$$L = n_1 = n_2 = \cdots = n_r > n_{r+1} \geqq \cdots \geqq n_s, \tag{157}$$

$$\sigma_1 = \sigma_2 = \cdots = \sigma_r. \tag{158}$$

Then from (154) we have

$$P_1 = P_2 = \cdots = P_r,$$

and it follows from (155) that

$$\frac{Z^{(k)} - \sigma N^{(k)}}{k^{L-1}} (\sigma_1 - \sigma)^{2k + L - 1} \to \frac{(-2)^{L-1}}{[(L-1)!]^2 P_1^2} \sum_{\nu=1}^{r} \eta_{2,\nu}^{(0)} U_L^{L-1} \xi_{2,\nu}^{(0)\prime} \qquad (k \to \infty). \tag{159}$$

If now we assume that

$$p_3 \equiv \sum_{\nu=1}^{r} \eta^{(0)}_{2,\nu}\, U_L^{L-1}\, \xi^{(0)\prime}_{2,\nu} \neq 0, \tag{160}$$

and this is an algebraic condition imposed upon ξ_0 and η_0, then we have

$$\begin{gathered} Z^{(k)} - \sigma N^{(k)} \sim \gamma_1 \frac{k^{L-1}}{(\sigma_1-\sigma)^{2k}} \qquad (k\to\infty), \\ \gamma_1 = \frac{(-2)^{L-1} p_3 (\sigma_1-\sigma)^{1-L}}{[(L-1)!]^2 P_1^2} \neq 0. \end{gathered} \tag{161}$$

On the other hand, it follows from (127) and (134), since $\delta < 1$, that the denominator of D_k in (98) is $\sim \eta_1^{(0)} \xi_1^{(0)\prime} \neq 0$, in virtue of (115). Therefore from (98) and (161) we obtain

$$D_k \sim \frac{\gamma_1}{\eta_1^{(0)}\, \xi_1^{(0)\prime}} \cdot \frac{k^{L-1}}{(\sigma_1-\sigma)^{2k}} \qquad (k\to\infty), \tag{162}$$

$$\frac{D_{k+1}}{D_k} \to \frac{1}{(\sigma_1-\sigma)^2} \qquad (k\to\infty) \tag{163}$$

and from (103) we have finally

$$\frac{\lambda_{k+1}-\sigma}{(\lambda_k-\sigma)^3} \to \frac{1}{(\sigma_1-\sigma)^2}\,. \tag{164}$$

43. We assume now that the conditions under B of Section 29 are satisfied, that is, that *σ and all σ_ν are real, and $N=1$*. Then, since each $n_\nu = 1$, (117) becomes

$$Z_\nu^{(k)} - \sigma N_\nu^{(k)} = (\eta^{(0)}_{2,\nu}\, \xi^{(0)\prime}_{2,\nu})\, (\sigma_\nu - \sigma) \prod_{\varkappa=0}^{k-1} (\sigma_\nu - \lambda_\varkappa)^{-2}. \tag{165}$$

For the product on the right, using the decomposition

$$\frac{1}{\sigma_\nu-\lambda_\varkappa} = \frac{1}{\sigma_\nu-\sigma}\left(1 - \frac{\sigma-\lambda_\varkappa}{\sigma_\nu-\lambda_\varkappa}\right),$$

we can write

$$\begin{gathered} \prod_{\varkappa=0}^{k-1} (\sigma_\nu - \lambda_\varkappa)^{-2} = (\sigma_\nu-\sigma)^{-2k}\, Q_{\nu,k}, \\ Q_{\nu,k} = \prod_{\varkappa=0}^{k-1} \left(1 - \frac{\sigma-\lambda_\varkappa}{\sigma_\nu-\lambda_\varkappa}\right)^2, \end{gathered} \tag{166}$$

and therefore

$$Z_\nu^{(k)} - \sigma N_\nu^{(k)} = (\eta^{(0)}_{2,\nu}\, \xi^{(0)\prime}_{2,\nu})\, (\sigma_\nu - \sigma)^{1-2k}\, Q_{\nu,k}. \tag{167}$$

From (166) we have

$$Q_{\nu,k+1} - Q_{\nu,k} = Q_{\nu,k}\left[\left(1 - \frac{\sigma-\lambda_k}{\sigma_\nu-\lambda_k}\right)^2 - 1\right] = Q_{\nu,k}\,\frac{\sigma-\lambda_k}{\lambda_k-\sigma_\nu}\left(\frac{\sigma_\nu-\sigma}{\sigma_\nu-\lambda_k}+1\right),$$

$$|Q_{\nu,k+1} - Q_{\nu,k}| \leqq \left|\frac{Q_{\nu,k}}{\sigma_\nu-\sigma_k}\left(\frac{\sigma_\nu-\sigma}{\sigma_\nu-\lambda_k}+1\right)\right|\, |\sigma-\lambda_\varkappa|\,. \tag{168}$$

44. Now, if the conditions (79) and (80) of Section 29 are satisfied, we have (135), and therefore the series $\sum_\varkappa |\sigma-\lambda_\varkappa|$ is convergent; further, $\sigma_\nu - \lambda_\varkappa$ tends to $\sigma_\nu - \sigma \neq 0$ and the series $\sum_\varkappa \left|\frac{\sigma-\lambda_\varkappa}{\sigma_\nu-\lambda_\varkappa}\right|$ is also convergent. We see that the product

$Q_{\nu,k}$ tends to a limit Q_ν as $k \to \infty$ and therefore the modulus of the coefficient of $|\sigma - \lambda_\varkappa|$ in (168) is not greater than a constant c independent of k and ν. From (168) we have

$$|Q_{\nu,k+1} - Q_{\nu,k}| \leqq c\,|\sigma - \lambda_\varkappa|,$$

and it follows that

$$|Q_\nu - Q_{\nu,k}| = \Bigl|\sum_{\varkappa=k}^{\infty} (Q_{\nu,\varkappa+1} - Q_{\nu,\varkappa})\Bigr| \leqq c \sum_{\varkappa=k}^{\infty} |\sigma - \lambda_\varkappa|$$

$$= c\,|\sigma - \lambda_k| \left(1 + \sum_{\varkappa>k} \left|\frac{\sigma - \lambda_\varkappa}{\sigma - \lambda_k}\right|\right) < 2c\,|\sigma - \lambda_k|$$

from some k on. We can therefore write (167) in the form

$$Z_\nu^{(k)} - \sigma N_\nu^{(k)} = (\eta_{2,\nu}^{(0)}, \xi_{2,\nu}^{(0)\prime})\,(\sigma_\nu - \sigma)^{1-2k}\, Q_\nu + O\bigl[(\sigma_\nu - \sigma)^{-2k}\,|\sigma - \lambda_k|\bigr], \tag{169}$$

$$Q_\nu = \prod_{\varkappa=0}^{\infty} \left(1 - \frac{\sigma - \lambda_\varkappa}{\sigma_\nu - \lambda_\varkappa}\right)^2. \tag{170}$$

Denoting the factor $(\eta_{2,\nu}^{(0)}, \xi_{2,\nu}^{(0)\prime})\,(\sigma_\nu - \sigma)\,Q_\nu$ by c_ν, we obtain

$$Z_\nu^{(k)} - \sigma N_\nu^{(k)} = \frac{c_\nu}{(\sigma_\nu - \sigma)^{2k}} + O\left(\frac{\sigma - \lambda_\varkappa}{(\sigma_\nu - \sigma)^{2k}}\right). \tag{171}$$

Put

$$\operatorname*{Max}_{\nu} \frac{1}{(\sigma_\nu - \sigma)^2} = \gamma. \tag{172}$$

Then by (102)

$$Z^{(k)} - \sigma N^{(k)} = \sum_{\nu=1}^{l} \frac{c_\nu}{(\sigma_\nu - \sigma)^{2k}} + O\bigl(|\sigma - \lambda_k|\,\gamma^k\bigr). \tag{173}$$

Denote the *different* values of the expression $\frac{1}{(\sigma_\nu - \sigma)^2}$ in decreasing order by

$$\gamma_1 > \gamma_2 > \cdots > \gamma_r.$$

Then (173) becomes

$$Z^{(k)} - \sigma N^{(k)} = s_1 \gamma_1^k + s_2 \gamma_2^k + \cdots + s_r \gamma_r^k + O\bigl(\gamma^k\,|\sigma - \lambda_k|\bigr). \tag{174}$$

Here some of the s_ϱ might vanish. If we retain only those γ_ϱ to which correspond non-vanishing s_ϱ and denote them in decreasing order by $g_1 > g_2 > \cdots > g_q$ and likewise the corresponding s_ϱ by $t_1, t_2, \ldots, t_q$, we obtain finally

$$Z^{(k)} - \sigma N^{(k)} = t_1 g_1^k + \cdots + t_q g_q^k + O\bigl(\gamma^k\,|\sigma - \lambda_k|\bigr), \tag{175}$$

unless all of the s_ϱ *vanish,* in which case we would get

$$Z^{(k)} - \sigma N^{(k)} = O\bigl(\gamma^k\,|\sigma - \lambda_k|\bigr). \tag{176}$$

45. If we assume that (176) is impossible, then (83) follows easily. Indeed, from (127) and (134) (observe that (112) can be assumed to be true)

$$|M_{k-1}^2 N^{(k)}| \leqq p_2^{(0)}\,\delta^2 (N^2 \delta^{k-1})^k \to 0,$$

and we see that the denominator in the expression (98) for D_k tends to $2p_1 \neq 0$ in virtue of (115). Therefore it follows from (98) and (175) that

$$D_k \sim \frac{t_1}{2p_1}\, g_1^k, \qquad \frac{D_{k+1}}{D_k} \sim g_1,$$

and by (103) we have
$$\frac{\lambda_{k+1}-\sigma}{(\lambda_k-\sigma)^3}=\frac{D_{k+1}}{D_k}\to g_1,$$
where $g_1=g$ is one of the numbers $\dfrac{1}{(\sigma_\nu-\sigma)^2}$.

It remains now to prove that (176) is impossible. Indeed, from (176) it would follow by (98), since the denominator tends to $2p_1$, that
$$D_k=O\left(\gamma^k|\sigma-\lambda_k|\right)$$
and by (97)
$$\lambda_k-\sigma=O\left(M_{k-1}^2\gamma^k|\sigma-\lambda_k|\right),$$
that is to say, that $\dfrac{1}{M_{k-1}^2\gamma^k}$ would be bounded. But by (134)
$$\frac{1}{\delta^{k^2-k+2}\gamma^k}=\frac{1}{\delta^2}\,\frac{1}{(\gamma\,\delta^{k-1})^k}$$
would be bounded too, while, since $0<\delta<1$, the denominator here tends to zero.

Therefore (176) is impossible, our assertion (83) is now proved in the Case B, and the proof of Theorem 3 is now finished.

46. We now discuss the convergence situation in an example. Consider the matrix

(177)
$$A=\begin{pmatrix}4 & 2\\ -3 & -1\end{pmatrix},$$

which has the eigenvalues 1 and 2. The eigenvectors corresponding to the eigenvalue 2 are $\xi=(1,-1)$ and $\eta=(1,\frac{2}{3})$.

We shall discuss the convergence of λ_k to 2. Put

(178)
$$\lambda_\varkappa=2+\mu_\varkappa,\qquad \xi_\varkappa=c_k(1,x_\varkappa),\qquad \lambda_\varkappa=c'_k(1,y_\varkappa),$$

(179)
$$x_\varkappa=-1+\sigma_\varkappa,\qquad y_\varkappa=\tfrac{2}{3}-\varrho_\varkappa.$$

Then we have
$$\begin{aligned}R(\xi_\varkappa,\eta_\varkappa)&=\frac{4-x_\varkappa y_\varkappa+2x_\varkappa-3y_\varkappa}{1+x_\varkappa y_\varkappa}\\&=\frac{4+(1-\sigma_\varkappa)\left(\frac{2}{3}-\varrho_\varkappa\right)-2(1-\sigma_\varkappa)-3\left(\frac{2}{3}-\varrho_\varkappa\right)}{1-(1-\sigma_\varkappa)\left(\frac{2}{3}-\varrho_\varkappa\right)}\\&=\frac{2+4\sigma_\varkappa+6\varrho_\varkappa+3\sigma_\varkappa\varrho_\varkappa}{1+2\sigma_\varkappa+3\varrho_\varkappa-3\sigma_\varkappa\varrho_\varkappa},\end{aligned}$$

(180)
$$\mu_\varkappa=R(\xi_\varkappa,\eta_\varkappa)-2=\frac{9\sigma_\varkappa\varrho_\varkappa}{1+2\sigma_\varkappa+3\varrho_\varkappa-3\sigma_\varkappa\varrho_\varkappa}.$$

Further, from (78), using (178) we have
$$x_\varkappa=-\frac{3+(3+\mu_\varkappa)\,x_{\varkappa+1}}{2-\mu_\varkappa+2x_{\varkappa+1}}$$
or
$$x_{\varkappa+1}=\frac{3+(2-\mu_\varkappa)\,x_\varkappa}{-(3+\mu_\varkappa)-2x_\varkappa}$$

and, using (179),

$$\sigma_{\varkappa+1} = \frac{3-(2-\mu_\varkappa)(1-\sigma_\varkappa)}{-3-\mu_\varkappa-2(1-\sigma_\varkappa)} + 1,$$

$$\sigma_{\varkappa+1} = \frac{\sigma_\varkappa \mu_\varkappa}{1+\mu_\varkappa+2\sigma_\varkappa}. \tag{181}$$

In the same way,

$$y_\varkappa = \frac{2-(3+\mu_\varkappa)\,y_{\varkappa+1}}{2-\mu_\varkappa-3\,y_{\varkappa+1}},$$

$$y_{\varkappa+1} = \frac{-2+(2-\mu_\varkappa)\,y_\varkappa}{-(3+\mu_\varkappa)+3\,y_\varkappa},$$

and again using (179),

$$\varrho_{\varkappa+1} = \frac{\varrho_\varkappa \mu_\varkappa}{1+\mu_\varkappa+3\,\varrho_\varkappa}. \tag{182}$$

47. If we start with $\xi_0=\eta_0=(1,1)$, $\sigma_0=2$, $\varrho_0=-\frac{1}{3}$, from (180) we have $\mu_0=\dfrac{-6}{1+4-1+2}=-1$, $\lambda_0=1$; here at the first step we get an eigenvalue, although ξ_0 and η_0 are not eigenvectors at all.

If we put

$$\xi_0=\eta_0=(1,0), \qquad x_0=y_0=0, \qquad \sigma_0=1, \qquad \varrho_0=\tfrac{2}{3},$$

we obtain from (180), (181) and (182) the values of $\sigma_\varkappa, \varrho_\varkappa, \mu_\varkappa$:

$$\sigma_1 = .4; \qquad \varrho_1 = .266666667; \qquad \mu_1 = .42105263,$$

$$\sigma_2 = .07582938; \qquad \varrho_2 = .050552923; \qquad \mu_2 = .026707002,$$

$$\sigma_3 = .0017186302; \qquad \varrho_3 = .0011457538; \qquad \mu_3 = .0^4 17601246,$$

$$\sigma_4 = .0^9 30145883; \qquad \varrho_4 = .0^7 2009726; \qquad \mu_4 = .0^{14} 5452646.$$

If we compute the quotients

$$\frac{\lambda_{\varkappa+1}-\sigma}{(\lambda_\varkappa-\sigma)^3} = \frac{\mu_{\varkappa+1}}{\mu_\varkappa^3}$$

for $\varkappa=1, 2, 3$ we obtain the following values taken to 4 significant figures:

$$.3579; \quad .9237; \quad 1.$$

If we now apply to the sequence $\lambda_\varkappa$ the argument in Section 19 Part II of this paper, we can expect an improvement upon λ_4 and μ_4 if the formula (61) is used with $\varkappa=3$. Thus we find that

$$\lambda_4-\lambda_3 = -.0^4 176\ldots, \qquad (\lambda_4-\lambda_3)^4 = .0^{19} 95278\ldots$$

$$\lambda_4-\lambda_2 = -.026707\ldots, \qquad (\lambda_4-\lambda_2)^3 = -.0^4 19049\ldots$$

$$\lambda_4'-\lambda_4 = -\frac{.0^{19} 95278}{.0^4 19049} = -.0^{14} 500178, \qquad \lambda_4' = 2.0^{15} 545\ldots$$

which is better by one decimal than $\lambda_4 = 2.0^{14} 545\ldots$.

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On the Convergence of the Rayleigh Quotient Iteration for the Computation of the Characteristic Roots and Vectors. IV

(Generalized Rayleigh Quotient for Nonlinear Elementary Divisors)

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48. In this part* we shall give a quadratically convergent iteration rule for computing an eigenvalue of a matrix to which there corresponds a non-linear elementary divisor. Our method generalizes the rule discussed in Sections 1—8, Part I, to non-symmetric matrices and combines it with rules for hastening the convergence of an iteration of the first order.

Let A be an $n \times n$ matrix with an eigenvalue σ, and assume that the maximal exponent L of an elementary divisor of A corresponding to σ, is >1. Choose two (row) vectors, α and β, and consider for a λ_0 which is different from all eigenvalues of A the linear equations

$$(A - \lambda_0 I)\,\xi' = \alpha', \quad \eta(A - \lambda_0 I) = \beta, \tag{183}$$

defining a couple of vectors ξ, η.

Now form the generalized Rayleigh quotient of ξ and η as defined by (72), and put

$$\lambda_1 = R(\xi, \eta) = \frac{\eta A \xi'}{\eta \xi'}, \tag{184}$$

assuming that

$$\eta \xi' \neq 0. \tag{184°}$$

Then from (184) and (183) we obtain λ_1 as a rational function of λ_0,

$$\lambda_1 = \varphi(\lambda_0). \tag{185}$$

We shall have to prove first that, under suitable hypotheses, the iteration by the function $\varphi(\lambda)$ defined by (185) has σ as a point of attraction. This is a consequence of the following

Theorem. *In the notation and under the assumptions of Section* 48, *there exists a matrix H depending only on A and on the choice of σ, such that if α and β are*

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chosen to satisfy

$$\beta H \alpha' \neq 0, \tag{186}$$

then we have

$$\varphi'(\sigma) = 1 - \frac{1}{L}. \tag{187}$$

In particular, if (186) *is satisfied, then the condition* (184°) *is also satisfied when* λ_0 *is sufficiently near to* σ.

49. (187) shows that the iteration by the function $\varphi(\lambda)$ converges rather slowly, though linearly; even for $L=2$ the derivative is $\frac{1}{2}$. However, since $\varphi(\lambda)$ is a rational function, it follows from (187) that, as soon as (186) is satisfied, we have a development

$$\varphi(\lambda) = \sigma + \left(1 - \frac{1}{L}\right)(\lambda - \sigma) + \sum_{\nu=2}^{\infty} a_\nu (\lambda - \sigma)^\nu, \tag{188}$$

and therefore different methods of acceleration are applicable.

Consider for an $\alpha \neq 1$ the function

$$\varphi^*(\lambda) = \frac{1}{1-\alpha}\left(\varphi(\lambda) - \alpha\lambda\right). \tag{189}$$

We verify at once that $\varphi^*(\sigma) = \sigma$ and obtain for the value of the derivative of φ^* in σ:

$$\varphi^{*\prime}(\sigma) = \frac{1 - \frac{1}{L} - \alpha}{1-\alpha}. \tag{190}$$

If now the value of L is known, we can choose $\alpha = 1 - \frac{1}{L}$ and obtain for the corresponding function φ^*:

$$\varphi_L(\lambda) = L\,\varphi(\lambda) - (L-1)\,\lambda \tag{191}$$

with vanishing derivative at the point σ. *The iteration by means of this function then converges quadratically to the value* σ.

If the value of L is not known, then it is still best to use the iterating function

$$\varphi_2(\lambda) = 2\varphi(\lambda) - \lambda. \tag{192}$$

The iteration by this function converges quadratically for $L=2$, while in the case of a general L we obtain a derivative

$$\varphi_2'(\sigma) = 1 - \frac{2}{L} \tag{193}$$

which is less than $1 - \frac{1}{L} = \varphi'(\sigma)$. On the other hand, if $L > 2$, the value of L will usually be recognized easily after a certain number of steps, and then the corresponding function $\varphi_L(\lambda)$ can be formed.

The use of $\varphi_2(\lambda)$ is best under the assumption that the value $L=2$ is the most probable of all $L>1$. On the other hand, if all values of L between 2 and n are more or less equally probable, we shall take

$$\alpha = \frac{1 - \frac{1}{n} + \frac{1}{2}}{2}, \tag{194}$$

and obtain readily

$$|\varphi^{*\prime}(\sigma)| \leqq \frac{1-\frac{2}{n}}{1+\frac{2}{n}}\,. \tag{195}$$

50. An iteration with quadratic convergence can be obtained in our case, if L is not known, by the Steffensen method, which, however, implies a double amount of computational work, for it uses a combination of λ, $\varphi(\lambda)$ and $\varphi\big(\varphi(\lambda)\big)$. In this method we form the function

$$\Phi(\lambda) = \frac{\varphi(\varphi(\lambda)) - \varphi(\lambda)^2}{\lambda - 2\varphi(\lambda) + \varphi(\varphi(\lambda))}\,, \tag{196}$$

which for $\lambda = \lambda_0$ usually gives a better approximation than

$$\lambda_2 = \varphi(\lambda_1) = \varphi\big(\varphi(\lambda_0)\big)\,.$$

In practice the value of $\varphi(\lambda_0)$ is of course computed by obtaining successively the values of λ_1 and λ_2 and by taking

$$\Phi(\lambda_0) = \frac{\lambda_0\,\lambda_2 - \lambda_1^2}{\lambda_0 - 2\,\lambda_1 + \lambda_2}\,. \tag{197}$$

If then we put

$$\lambda_1' = \Phi(\lambda_0')\,, \qquad \lambda_0' = \lambda_0\,, \tag{198}$$

we can consider the passage from λ_0' to λ_1' as a direct iteration by the iterating function $\Phi(\lambda)$.

However, this iteration converges quadratically. We prove this by using the result of our paper*. If we replace there λ_1 and λ_2 by 2, y by λ, ζ by σ and $\alpha_1 = \alpha_2$ by $1 - \frac{1}{L}$, we obtain from formula (21) *l.c.*

$$\frac{\Phi(\lambda) - \sigma}{(\lambda - \sigma)^2} = L^2\, T_z + O(\lambda - \sigma)$$

where T_z is obtained from the formulae (15) and (9) *l.c.*:

$$T_z = \left(1 - \frac{1}{L}\right)^2 E\big(\varphi(\lambda)\big) - \left(1 - \frac{1}{L}\right) E(\lambda)\,,$$

$$E(\lambda) = a_2 + a_3(\lambda - \sigma) + \cdots,$$

$$T_z = \left(1 - \frac{1}{L}\right)^2 a_2 - \left(1 - \frac{1}{L}\right) a_2 + O(\lambda - \sigma)$$

$$= -\frac{1}{L}\left(1 - \frac{1}{L}\right) a_2 + O(\lambda - \sigma)\,,$$

$$\frac{\Phi(\lambda) - \sigma}{(\lambda - \sigma)^2} = -(L - 1)\, a_2 + O(\lambda - \sigma)\,. \tag{199}$$

The one theoretically unsurmountable difficulty in the practical application of this method appears to be the fact that the bilinear form in (186) is unknown so long as the transformation of A to the Jordan canonical form has not been carried out, and this presupposes the knowledge of the eigenvalues. However, in practice this is hardly a difficulty at all, since it is infinitely improbable that for α and β taken at random, (186) would not be satisfied.

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* Ostrowski, A.: Über Verfahren von Steffensen und Householder zur Konvergenzverbesserung von Iterationen [Mauro Picone zum 70. Geburtstag, ZAMP Vol. VII, 218–229 (1956)].

51. Lemma 1. *Consider the matrix*

$$A_0 = \sigma I + U, \tag{200}$$

I being the unity matrix of order l and U the corresponding auxiliary unity matrix of order l, which has 1's in the first superdiagonal and zeros elsewhere. Then we have, for $\lambda \neq \sigma$, $\lambda \to \sigma$:

$$(A_0 - \lambda I)^{-2} = l \frac{U^{l-1}}{(\lambda-\sigma)^{l+1}} + O\left(\frac{1}{(\lambda-\sigma)^l}\right), \tag{201}$$

$$U(A_0 - \lambda I)^{-2} = (l-1) \frac{U^{l-1}}{(\lambda-\sigma)^{l}} + O\left(\frac{1}{(\lambda-\sigma)^{l-1}}\right). \tag{202}$$

Proof. Putting $\varkappa = \frac{1}{\lambda-\sigma}$ we have, since $U^l = 0$,

$$\begin{aligned}((\sigma-\lambda) I + U)^{-2} &= \varkappa^2 (I - \varkappa U)^{-2} \\ &= \sum_{\nu=0}^{l-1} (\nu+1) \varkappa^{\nu+2} U^\nu,\end{aligned}$$

and, multiplying this by U,

$$U((\sigma-\lambda) I + U)^{-2} = \sum_{\nu=0}^{l-2} (\nu+1) \varkappa^{\nu+2} U^{\nu+1},$$

as $U^l = 0$. Taking out the highest terms on the right, we obtain (201) and (202).

52. Lemma 2. *Let σ be an eigenvalue of the matrix A to which correspond elementary divisors with the maximal exponent $L > 1$. Then there exists a matrix $H \neq 0$ such that we have for $\lambda \to \sigma$:*

$$(A - \lambda I)^{-2} = L \frac{H}{(\lambda-\sigma)^{L+1}} + O\left(\frac{1}{(\lambda-\sigma)^L}\right), \tag{203}$$

$$A(A-\lambda I)^{-2} - \sigma (A - \lambda I)^{-2} = (L-1) \frac{H}{(\lambda-\sigma)^L} + O\left(\frac{1}{(\lambda-\sigma)^{L-1}}\right). \tag{204}$$

53. Proof. Since the assertions (203) and (204) are invariant with respect to a similarity transformation, though H does change, we can assume from the beginning that A is given in the Jordan canonical form. We can therefore write A as a direct sum,

$$A = \sum_i{}^{\boldsymbol{\cdot}} A_i \dotplus B, \tag{205}$$

where B is a matrix with eigenvalues $\neq \sigma$, while each A_i is a matrix of order m_i given by

$$A_i = \sigma I_{m_i} + U_{m_i} \qquad (i = 1, \ldots, k), \; L = \max_i m_i, \tag{206}$$

I_{m_i} and U_{m_i} having meanings analogous to those of I and U in Lemma 1.

54. Then for $\lambda \to \sigma$ we obviously have

$$(B - \lambda I)^{-2} = O(1),$$

and therefore

$$(A - \lambda I)^{-2} = \sum_{i=1}^{k}{}^{\cdot}\, (A_i - \lambda I_{m_i})^{-2} \dotplus O(1).$$

Applying the result of Lemma 1 to each of the first k terms in the direct sum on the right, we obtain

$$(A - \lambda I)^{-2} = L \, \frac{\sum\limits_{i}{}^{\cdot}\, U_{m_i}^{L-1}}{(\lambda - \sigma)^{L+1}} + O\left(\frac{1}{(\lambda - \sigma)^{L}}\right).$$

Here in the direct sum

$$H = \sum_{i}{}^{\cdot}\, U_{m_i}^{L-1} \tag{207}$$

we must sum over all i with $m_i = L$. However, for $m_i < L$ we have $U_{m_i}^{L-1} = 0$, and therefore we can write

$$H = \sum_{i=1}^{k}{}^{\cdot}\, U_{m_i}^{L-1},$$

where $H \neq 0$, since there exist $m_i = L$.

This proves (203).

55. Further, from (205) we have

$$A(A - \lambda I)^{-2} - \sigma (A - \lambda I)^{-2} = \sum_{i=1}^{k}{}^{\cdot}\, (A_i - \sigma I_{m_i})\, (A - \lambda I_{m_i})^{-2} \dotplus O(1).$$

Applying (202) to the first k terms in the sum on the right, we have now

$$(A_i - \sigma I_{m_i})\, (A_i - \lambda I_{m_i})^{-2} = U_{m_i} (A_i - \lambda I_i)^{-2} = (m_i - 1)\, \frac{U_{m_i}^{m_i - 1}}{(\lambda - \sigma)^{m_i}} + O\left(\frac{1}{(\lambda - \sigma)^{m_i - 1}}\right).$$

Therefore

$$A(A - \lambda I)^{-2} - \sigma (A - \lambda I)^{-2} = (L - 1)\, \frac{\sum\limits_{i} U_{m_i}^{L-1}}{(\lambda - \sigma)^{L}} + O\left(\frac{1}{(\lambda - \sigma)^{L-1}}\right),$$

where the numerator of the first fraction on the right is given by (207). This proves (204).

56. Proof of the theorem of Section 48. Take the matrix H occurring in the formulae (203), (204) of Lemma 2 and assume that the condition (186) is satisfied. Then from (184) and (185) for $\lambda_0 \to \sigma$ we have

$$\varphi(\lambda_0) - \sigma = \frac{\eta A \xi' - \sigma (\eta \xi')}{\eta \xi} = \frac{\beta A (A - \lambda I)^{-2} \alpha' - \sigma \beta (A - \lambda_0 I)^{-2} \alpha'}{\beta (A - \lambda_0 I)^{-2} \alpha'}.$$

From (203) and (204)

$$\varphi(\lambda_0) - \sigma = \frac{(L - 1)\, \dfrac{\beta H \alpha'}{(\lambda_0 - \sigma)^{L}} + O\left(\dfrac{1}{(\lambda_0 - \sigma)^{L-1}}\right)}{L\, \dfrac{\beta H \alpha'}{(\lambda_0 - \sigma)^{L+1}} + O\left(\dfrac{1}{(\lambda_0 - \sigma)^{L}}\right)} = (\lambda_0 - \sigma)\, \frac{L - 1 + O(\lambda_0 - \sigma)}{L + O(\lambda_0 - \sigma)}.$$

This proves assertion (187) of the theorem. The last part of the theorem follows immediately by (203) from

$$\eta \xi' = \beta (A - \lambda_0 I)^{-2} \alpha' = L \frac{\beta H \alpha'}{(\lambda_0 - \sigma)^{L+1}} + O\left(\frac{1}{(\lambda_0 - \sigma)^{L}}\right).$$

57. We consider as an example the matrix

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 3 \\ 1 & -1 & 1 \end{pmatrix}.$$

Here $(A - \lambda I)$ has the elementary divisors $(\lambda - 1)^2$ and $(\lambda - 2)$, and we have

$$D = |A - \lambda I| = (\lambda - 1)^2 (2 - \lambda) = -(\lambda^3 - 4\lambda^2 + 5\lambda - 2).$$

We easily obtain

$$D(A - \lambda I)^{-1} = \begin{pmatrix} \lambda^2 - 2\lambda + 4, & -3, & 3\lambda - 3 \\ \lambda + 2, & \lambda^2 - 3\lambda - 1, & 3\lambda - 3 \\ \lambda - 2, & 2 - \lambda, & \lambda^2 - 3\lambda + 2 \end{pmatrix},$$

$$\begin{aligned} X &= \frac{D^2}{\lambda - 1} (A - \lambda I)^{-2} \\ &= \begin{pmatrix} \lambda^3 - 3\lambda^2 + 12\lambda - 16, & -9\lambda + 15, & 6\lambda^2 - 15\lambda + 9 \\ 2\lambda^2 + 4\lambda - 12, & \lambda^3 - 5\lambda^2 - \lambda + 11, & 6\lambda^2 - 15\lambda + 9 \\ 2\lambda^2 - 8\lambda + 8, & -2\lambda^2 + 8\lambda - 8, & \lambda^3 - 5\lambda^2 + 8\lambda - 4 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} Y &= \frac{D^2}{\lambda - 1} A (A - \lambda I)^{-2} \\ &= \begin{pmatrix} 2(\lambda^3 - 4), & -6(\lambda^2 - \lambda - 1), & 3(\lambda^3 - \lambda^2 - 2\lambda + 2) \\ \lambda^3 + 5\lambda^2 - 8\lambda - 4, & \lambda^3 - 11\lambda^2 + 14\lambda + 2, & 3(\lambda^3 - \lambda^2 - 2\lambda + 2) \\ \lambda^3 - 3\lambda^2 + 4, & -(\lambda^3 - 3\lambda^2 + 4), & \lambda^3 - 5\lambda^2 + 8\lambda - 4) \end{pmatrix}. \end{aligned}$$

From the definition in Section 48,

$$\varphi(\lambda) = \frac{\beta Y \alpha'}{\beta X \alpha'}. \tag{208}$$

Since we have

$$Y - X = (\lambda - 1) \begin{pmatrix} \lambda^2 + 4\lambda - 8 & -3(2\lambda - 3) & 3(\lambda - 1)^2 \\ \lambda^2 + 4\lambda - 8 & -3(2\lambda - 3) & 3(\lambda - 1)^2 \\ (\lambda - 2)^2 & -(\lambda - 2)^2 & 0 \end{pmatrix} = (\lambda - 1) Z,$$

it follows that

$$\frac{\Phi(\lambda) - 1}{\lambda - 1} = \frac{\beta Z \alpha'}{\beta X \alpha'}. \tag{209}$$

For $\lambda = 1$, Z and X become respectively

$$Z_0 = \begin{pmatrix} -3 & 3 & 0 \\ -3 & 3 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad X_0 = \begin{pmatrix} -6 & 6 & 0 \\ -6 & 6 & 0 \\ 2 & -2 & 0 \end{pmatrix} = 2 Z_0,$$

and we see that as $\lambda \to 1$ the ratio (209) tends to $\frac{1}{2}$ if $\beta Z_0 \alpha' \neq 0$. Z_0 is therefore the matrix H in the condition (186), and if we write

$$\alpha = (a_1, a_2, a_3), \quad \beta = (b_1, b_2, b_3),$$

this condition becomes

$$(a_1 - a_2)(-3b_1 - 3b_2 + b_3) \neq 0.$$

We see that α and β have to be chosen in such a way that $a_1 \neq a_2$, $b_3 \neq 3(b_1 + b_2)$.

If, for instance, we take

$$\alpha = (0, 1, 0), \quad \beta = (1, 0, 0),$$

we have from (208)

$$\varphi(\lambda) = 2\,\frac{\lambda^2 - \lambda - 1}{3\lambda - 5} \tag{210}$$

and from (192)

$$\varphi_2(\lambda) = \frac{\lambda^2 + \lambda - 4}{3\lambda - 5}. \tag{211}$$

If we start with $\lambda_0 = 0$ we obtain from (210) the following values of λ_ν and $\frac{1-\lambda_\nu}{1-\lambda_{\nu-1}}$:

ν	λ_ν	$\frac{1-\lambda_\nu}{1-\lambda_{\nu-1}}$
0	0.	
1	.4	.6
2	.6526	.579
3	.80648	.557
4	.89599	.5375
5	.94565	.5225
6	.97214	.5126

Starting again with $\lambda_0 = 0$ we obtain from (211)

ν	λ_ν	$\frac{(1-\lambda_{\nu-1})^2}{1-\lambda_\nu}$
0	0	
1	.8	5
2	.984616	2.046
3	.999884338	$2.0^{3}347$
4	$.9^{8}331229$	$2.0^{7}2006$

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Characterizations of Laplace Transforms and Perfect Operators

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1. The notion of a perfect operator was introduced in [*1*], mainly for the purpose of widening the Laplace-transform interpretation of HEAVISIDE'S symbolic calculus. A problem mentioned in [*1*], and related to the classical problem of characterizing Laplace transforms, is that of finding a necessary and sufficient condition for an analytic function to be the Laplace transform of a perfect operator. We give here a solution of this problem, in a way that enables us to show (in answer to a question raised in [*2*]) that all perfect operators are obtainable by repeated "differentiation" of continuous functions. We also discuss some implications of these results. The notation and terminology are those of [*2*], supplemented where necessary.

2. We shall require a pair of lemmata concerning Laplace transforms of functions: we give proofs of these for the sake of completeness, though in fact much stronger results are known.

Lemma 1. *Let x be a continuous function of a positive real variable, and suppose that, for some real number a, $e^{-at}x(t)$ is bounded when $t>0$. Let $\bar{x}$ be the Laplace transform of x, and let c be a real number greater than a. Then $\bar{x}(z)$ is bounded when $\mathscr{R}z \geqq c$.*

Proof. When $\mathscr{R}z \geqq c$,

$$\bar{x}(z) = \int_0^\infty e^{-zt}\, x(t)\, dt,$$

so that, if $|e^{-at}x(t)| \leqq k$ when $t>0$,

$$|\bar{x}(z)| \leqq \int_0^\infty |e^{-zt}\, x(t)|\, dt \leqq \int_0^\infty k\, e^{(a-c)t}\, dt = \frac{k}{c-a}.$$

Lemma 2. *Let g be a function which is holomorphic in a domain containing a closed half-plane $\mathscr{R}z \geqq c$, and suppose that $z^2 g(z)$ is bounded when $\mathscr{R}z \geqq c$. Then the formula*

$$x(t) = \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} g(c+i\omega)\, e^{i\omega t}\, d\omega \qquad (t>0)$$

defines a continuous function x such that $e^{-ct}x(t)$ is bounded when $t>0$, and $\bar{x}(z) = g(z)$ when $\mathscr{R}z > c$.

Proof. Since $(c+i\omega)^2 g(c+i\omega)$ is bounded when ω is real,

$$|e^{-ct} x(t)| \leqq \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(c+i\omega)|\, d\omega < \infty;$$

and there is no difficulty in showing that x is a continuous function. If $\mathscr{R} z > c$, then

$$\bar{x}(z) = \frac{1}{2\pi} \int_0^{\infty} e^{(c-z)t} \int_{-\infty}^{\infty} g(c+i\omega)\, e^{i\omega t}\, d\omega\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(c+i\omega) \int_0^{\infty} e^{(c+i\omega-z)t}\, dt\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{g(c+i\omega)}{c+i\omega-z}\, d\omega.$$

Now, for each positive number ϱ, let S_ϱ and C_ϱ be, respectively, the straight path and the right-hand semi-circular path from $c-i\varrho$ to $c+i\varrho$. By the residue theorem, if $\mathscr{R} z > c$ and $\varrho > |z-c|$,

$$2\pi i\, g(z) = \int_{S_\varrho} \frac{g(s)}{s-z}\, ds - \int_{C_\varrho} \frac{g(s)}{s-z}\, ds.$$

But, when s is on C_ϱ, $s g(s)$ is uniformly bounded with respect to ϱ; so that, by a standard argument,

$$\int_{C_\varrho} \frac{g(s)}{s-z}\, ds \to 0 \quad \text{as} \quad \varrho \to \infty.$$

Thus, when $\mathscr{R} z > c$,

$$g(z) = \lim_{\varrho \to \infty} \frac{1}{2\pi i} \int_{S_\varrho} \frac{g(s)}{s-z}\, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{g(c+i\omega)}{c+i\omega-z}\, d\omega = \bar{x}(z).$$

3. We recall that a perfect function, of a positive real variable, is a function p which has derivatives of all orders and is such that, for $n = 0, 1, 2, \ldots$, $p^{(n)}(t)$ is of exponential order (as $t \to \infty$) and $p^{(n)}(0+) = 0$. The following result gives a characterization of the Laplace transforms of perfect functions.

Theorem 1. *A function φ, holomorphic in a half-plane $\mathscr{R} z > a$, is the Laplace transform of a perfect function if and only if, for every integer n, there is a real number c_n such that $z^n \varphi(z)$ is bounded when $\mathscr{R} z \geqq c_n$.* *

Proof. Suppose first that φ is the Laplace transform of a perfect function p, and let n be any non-negative integer. If $x = p^{(n)}$ then $\bar{x}(z) = z^n \varphi(z)$, so that, by Lemma 1, there is a real number c_n such that $z^n \varphi(z)$ is bounded when $\mathscr{R} z \geqq c_n$. We may take c_n to be positive, so that $z^{-n} \varphi(z)$ is also bounded when $\mathscr{R} z \geqq c_n$.

Conversely, suppose that, for every integer n, there is a c_n such that $z^n \varphi(z)$ is bounded when $\mathscr{R} z \geqq c_n$. Then $z^{n+2} \varphi(z)$ is bounded when $\mathscr{R} z \geqq c_{n+2}$, so that, by Lemma 2 (with $g(z) = z^n \varphi(z)$), there is a continuous function x_n, of exponential order, such that

$$\overline{x_n}(z) = z^n \varphi(z).$$

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* It is not enough that $z^n f(z)$ should be bounded when $\mathscr{R} z = c_n$, for every n: this condition is satisfied if $f(z) = \Gamma(z)$, for example, but then $z^n f(z)$ is unbounded when $\mathscr{R} z \geqq c$, for every real c.

Now, since $\overline{x_n}(z) = z^{-1}\,\overline{x_{n+1}}(z)$,

$$x_n(t) = \int_0^t x_{n+1}(u)\,du \qquad (t > 0),$$

so that $x_n(0+) = 0$ and $x_{n+1} = x_n'$. Thus x_0 is a perfect function, with $x_0^{(n)} = x_n$ when $n \geqq 0$. Since $\varphi = \overline{x_0}$, the theorem follows.

4. Our main result is

Theorem 2. *A function f, holomorphic in a half-plane $\mathscr{R} z > a$, is the Laplace transform of a perfect operator if and only if, for some integer n, there is a real number c such that $z^n f(z)$ is bounded when $\mathscr{R} z \geqq c$. Every perfect operator is of the form $D^m X$, where D is the differential operator (on perfect functions), m is an integer, and X is the convolution operator determined by a continuous function.*

Proof. Suppose first that f is the Laplace transform of a perfect operator $f(D)$, and let us assume that, for every integer n and every real number c, $z^n f(z)$ is unbounded when $\mathscr{R} z \geqq c$. Then a sequence $z_1, z_2, z_3, \ldots$ can be defined inductively so that $\mathscr{R} z_1 \geqq e$ and, for $n = 1, 2, 3, \ldots$,

(i) $$|z_n^{-n} f(z_n)| \geqq n,$$

(ii) $$\mathscr{R}\, z_{n+1} \geqq e^n |z_n|^2.$$

Let $w_n = \log z_n$, for each n (the logarithms having their principal values). Then it follows from (ii) that $|w_{n+1}/w_n| \geqq 2$, and hence that the infinite products

$$\prod_{n=1}^{\infty}\left(1 - \frac{w}{w_n}\right), \qquad \prod_{n=1}^{\infty} e^{-2w/|w_n|}$$

converge for all values of the complex variable w, and represent entire functions. For each positive integer m, let ψ_m be the entire function defined by the formula

$$\psi_m(w) = \left(1 - \frac{w}{w_m}\right)^{-1} \prod_{n=1}^{\infty}\left(1 - \frac{w}{w_n}\right) e^{-2w/|w_n|}.$$

Then

(iii) $$\psi_m(w_n) = 0 \qquad (m \neq n)$$

and, for each m,

$$|\psi_m(w_m)| \geqq \left\{\prod_{n=1}^{m-1}\left(\left|\frac{w_m}{w_n}\right| - 1\right)\right\}\left\{\prod_{n=m+1}^{\infty}\left(1 - \left|\frac{w_m}{w_n}\right|\right)\right\} |e^{-\sigma w_m}|,$$

where $\sigma = 2\sum_{n=1}^{\infty} |w_n|^{-1}$. Now $|w_m/w_n| \geqq 2$ when $n \leqq m-1$, and $|w_m/w_n| \leqq 2^{m-n}$ when $n \geqq m+1$; hence

$$|\psi_m(w_m)| \geqq \left\{\prod_{n=m+1}^{\infty}(1 - 2^{m-n})\right\} |e^{-\sigma w_m}|.$$

Thus

(iv) $$|\psi_m(w_m)| \geqq \delta\, |z_m|^{-\sigma},$$

where $\delta = \prod_{n=1}^{\infty}(1 - 2^{-n})$, a positive number which, like σ, is independent of m.

With $\mathscr{R}z \geqq e$, let $w = \log z$. Then $\mathscr{R}w \geqq 1$ and $|\Im w| < \pi/2$; so that $2\,\mathscr{R}w \geqq |w|$. Hence, for each n,

$$\left|\left(1-\frac{w}{w_n}\right)e^{-2w/|w_n|}\right| \leqq \left(1+\left|\frac{w}{w_n}\right|\right)\left|e^{-2w/|w_n|}\right| \leqq \left(1+\left|\frac{w}{w_n}\right|\right)e^{-|w/w_n|} \leqq 1\,.$$

Since, for each m,

$$\psi_m(w) = e^{-2w/|w_m|}\prod_{n \neq m}\left(1-\frac{w}{w_n}\right)e^{-2w/|w_n|},$$

it follows that, when $\mathscr{R}z \geqq e$,

(v) $$|\psi_m(w)| \leqq 1\,.$$

Now, for $m = 1, 2, 3, \ldots$, and $\mathscr{R}z \geqq e$, let

$$\varphi_m(z) = e^{-\sqrt{(2z/|z_m|)}}\,\psi_m(\log z)\,.$$

Then each function φ_m is holomorphic in the half-plane $\mathscr{R}z > e$, and, for each non-negative integer n,

$$|z^n\,\varphi_m(z)| \leqq |z|^n\left|e^{-\sqrt{(2z/|z_m|)}}\right|\,|\psi_m(w)| \leqq |z|^n\left|e^{-\sqrt{(2z/|z_m|)}}\right|,$$

by (v). But, if $\tau = \sqrt{|z/z_m|}$,

$$|z|^n\left|e^{-\sqrt{(2z/|z_m|)}}\right| \leqq |z_m|^n\,\tau^{2n}\,e^{-\tau},$$

and $\tau^{2n}e^{-\tau}$, as a function of the positive variable τ, has the maximum value $(2n/e)^{2n}$, attained when $\tau = 2n$. Thus

(vi) $$|z^n\,\varphi_m(z)| \leqq n^{2n}\,|z_m|^n.$$

It is a consequence of (ii) that, for any integer n,

(vii) $$\sum_{m=1}^{\infty}|z_m|^{n+\sigma-m} < \infty\,.$$

Let

$$\varphi(z) = \sum_{m=1}^{\infty}|z_m|^{\sigma-m}\,\varphi_m(z)\,.$$

By (vi) and (vii), with $n = 0$, this series converges uniformly when $\mathscr{R}z \geqq e$; so that the function φ is holomorphic in the half-plane $\mathscr{R}z > e$. Moreover, when $n \geqq 0$,

$$|z^n\varphi(z)| \leqq n^{2n}\sum_{m=1}^{\infty}|z_m|^{n+\sigma-m}.$$

Thus, for every integer n, $z^n\varphi(z)$ is bounded when $\mathscr{R}z \geqq e$. By Theorem 1, therefore, φ is the Laplace transform of a perfect function, say p.

Let q be the perfect function $f(D)\,p$. Then, when $\mathscr{R}z$ is sufficiently large, $\bar{q}(z) = f(z)\,\varphi(z)$. In particular, for each positive integer m,

$$\bar{q}(z_m) = f(z_m)\,|z_m|^{\sigma-m}\,\varphi_m(z_m)\,,$$

since $\varphi_m(z_n)=0$ when $m \neq n$, by (iii). Hence, by (i) and (iv),

$$|\bar{q}(z_m)| \geqq m\,|z_m|^{\sigma}\,|e^{-\sqrt{(2 z_m/|z_m|)}}|\,\delta\,|z_m|^{-\sigma} \geqq m\,e^{-\sqrt{2}}\,\delta.$$

Thus $\bar{q}(z_m) \to \infty$ as $m \to \infty$. By (ii), therefore, $\bar{q}(z)$ is unbounded when $\mathscr{R} z \geqq c$, for any real number c. This contradicts Theorem 1, forcing us to conclude that, contrary to our initial assumption, $z^n f(z)$ is bounded when $\mathscr{R} z \geqq c$, for some integer n and some real number c.

Conversely, suppose f to be such that $z^n f(z)$ is bounded when $\mathscr{R} z \geqq c$. By Lemma 2 (with $g(z)=z^{n-2} f(z)$), there is a continuous function x, of exponential order, such that

$$\bar{x}(z) = z^{n-2} f(z).$$

Thus f is the Laplace transform of the perfect operator $D^m X$, where $m=2-n$ and X is the convolution operator determined by x. This completes the proof of Theorem 2.

5. In [*2*] it is shown that the perfect operators can be characterized as those linear operators, on perfect functions, which commute with the differential operator D and are continuous in a certain sense. We take this occasion to point out that not all linear operators that commute with D are perfect. To this end, we note that the perfect operators whose Laplace transforms are rational functions constitute a field, $\mathfrak{F}$, and that the perfect functions constitute a vector space over $\mathfrak{F}$, in an obvious way. This vector space is not one-dimensional, and therefore has a basis containing at least two perfect functions, p, q; and we can define a linear operator A on the space in such a way that $Ap=0$ and $Aq \neq 0$. Such an operator cannot be perfect, since the algebra of Laplace transforms has no divisors of zero. However, since A is linear with respect to $\mathfrak{F}$, it commutes with D (which belongs to $\mathfrak{F}$) and is linear with respect to the complex numbers (which are embedded in $\mathfrak{F}$, in the sense that $\mathfrak{F}$ contains the scalar multiples of the unit operator, I).

6. Systems of linear differential equations that occur in certain physical problems can be formulated in terms of perfect operators: all functions of the time variable are replaced by the corresponding perfect operators (which may then be taken to represent "generalized functions", including impulses if necessary), the symbol "dx/dt" being replaced by "$DX - x(0+)I$", and so forth. If a solution can then be found within the system of perfect operators, this will have a direct physical interpretation, since, by Theorem 2, every perfect operator represents an "m^{th} derivative" of a continuous function. Thus "spurious" solutions cannot occur. In the system of operational calculus which has been developed by MIKUSIŃSKI [*3*], solutions sometimes occur which have to be rejected on grounds that seem extraneous to the original mathematical framework. For example, in the discussion of a certain type of boundary-value problem, the expressions "$e^{\sqrt{D}}$" and "$e^{-\sqrt{D}}$" appear to have equal validity, although only the second has a physical meaning. In the system of perfect operators, on the other hand, it is only the second expression that has a mathematical meaning, since the function $e^{\sqrt{z}}$ does not satisfy the conditions of Theorem 2 (in fact it was shown in [*1*] that this function is not a Laplace transform).

In Mikusiński's system, the "operators" constitute a field; but it is easy to see that for some purposes a field is too large. For instance, the operator e^{-uD}, where $u > 0$, represents a "pure delay", so that, since there are functions which are not obtainable by delaying others, it is inappropriate that e^{-uD} should have a reciprocal in the system. It is of course possible to restrict attention to those elements of the field which represent generalized derivatives of continuous functions. In the system of perfect operators, this restriction is inherent; and there is also the advantage that each continuous function considered has a Laplace transform, an "operational form" of the function being obtainable by simply replacing the symbol for the complex variable, in an expression of the Laplace transform, by the symbol "D". Moreover, Theorem 2 provides a simple criterion for deciding whether a given analytic function is a Laplace transform or not; and if it is, we can "invert" it by "differentiating", a suitable number of times, a continuous function which is given by a Fourier integral according to Lemma 2. The price of these advantages, namely the condition "$x(t) = O(e^{ct})$ as $t \to \infty$", is not exorbitant.

The field $\mathfrak{F}$, consisting of all perfect operators that can be expressed rationally in terms of D, contains the solutions of all finite systems of ordinary linear differential equations in which the coefficients on the left-hand side are constant and the "functions" on the right-hand side have rational Laplace transforms. After formulating the equations in terms of perfect operators, the solutions can be found by the usual processes of rational algebra (which are valid for any field), and interpreted in terms of partial fractions.

7. The use of perfect operators in linear electric-circuit theory is closely analogous to the use of complex numbers to represent currents, voltages, impedances, and so on, in the steady-state sinusoidal theory with which engineers are familiar. The various "operational impedances" which determine the response of a network to any given excitation can be found as follows: first calculate the steady-state impedances in terms of a frequency parameter ν, and then substitute "$D/2\pi i$" for "ν" at each occurrence. The operational impedances of a finite network are always elements of the field $\mathfrak{F}$, and all the usual network theorems remain valid.

Differences of opinion have been sharply expressed, even in quite recent times, over the relative merits of the "operational" and the "transform" points of view in interpreting Heaviside's symbolism (see, for example, [*4*] and [*5*]). It may be hoped that these differences can be reconciled by appealing to the natural isomorphism (denoted in [*2*] by "Λ") between the algebra of perfect operators and the algebra of Laplace transforms. However, in the notation used by Heaviside, and still sometimes advocated, the "operational form" of the unit function is "1": this is unnatural from the operational point of view that we have been considering, since the perfect operator determined by the unit function is the integrator, D^{-1}, not the unit operator I (which represents a unit impulse at the instant $t = 0$). Our terminology could be altered to accommodate Heaviside's notation, with perhaps some practical advantage; but then the operators "determined" by functions would not form a ring, and their Laplace transforms would differ (by the factor z) from those of the functions themselves.

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The Cauchy Problem and the Mixed Boundary Value Problem for a Non-Linear Hyperbolic Partial Differential Equation in Two Independent Variables

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Preface

The following paper deals with the Cauchy problem and the mixed value problem for a canonical form of a second order partial differential equation of hyperbolic type in two independent variables. The exact statements of the Cauchy problem and of the mixed value problem dealt with are given in Theorems 1 and 8 respectively.

Theorem 1, which concerns the Cauchy problem, has been proved by P. LEEHEY [*1*], see also P. HARTMAN & A. WINTNER [*2*]. These two theorems are proved here by a variation of a method employed by J. B. DIAZ (to whom the author is indebted for his assistance in the preparation of this paper) in the treatment of the characteristic boundary value problem [*3*]. The main interest of the present approach is that the existence theorems are proved in a way which leads directly to a method for the actual numerical construction of solutions.

Appendix 1 is concerned with concrete numerical applications of the present method to the mixed boundary value problem and to the Cauchy problem. In order to make the paper self-contained, Appendix 2 presents a slight generalization of a relatively little known theorem of ARZELÀ, which is fundamental for the present method.

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Section I. The Cauchy Problem

1. *Statement of the Problem*

Section I of this paper deals with the Cauchy problem for a hyperbolic partial differential equation in two independent variables, the precise statement of the problem being given in the following theorem.

Theorem 1. *If:*

a) f is a real valued function of the five real variables x, y, z, p, q (this includes the case where f is independent of some or all of the variables x, y, z, p, q);

b) f is continuous for $0 \leqq x,\ y \leqq A$; $-\infty < z, p, q < \infty$;

c) There is a constant $M \geqq 0$ such that $|f(x, y, z, p, q)| \leqq M$ for $0 \leqq x,\ y \leqq A$; $-\infty < z, p, q < \infty$;

d) f satisfies a Lipschitz condition in p and q, i.e., *there is a constant $L > 0$ such that for $0 \leqq x,\ y \leqq A$ and $-\infty < z, p, q, p_0, q_0 < \infty$ we have $|f(x, y, z, p, q) - f(x, y, z, p_0, q_0)| \leqq L\{|p - p_0| + |q - q_0|\}$;*

e) σ and τ are continuous real valued functions of the real variable λ for $0 \leqq \lambda \leqq A$;

then there is at least one u such that for $0 \leqq x,\ y \leqq A$:

f) u is a function of x and y;

g) u and its partial derivatives u_x, u_y and $u_{xy} = u_{yx}$ are continuous;

h) $u_{xy}(x, y) = f[x, y, u(x, y), u_x(x, y), u_y(x, y)]$;

i) $u(0, 0) =$ constant (which can be taken to be zero);

j) $u_x(x, y)|_{y=x} = \sigma(x)$;

k) $u_y(x, y)|_{y=x} = \tau(x)$.

The hypotheses of Theorem 1 are not sufficient to guarantee the uniqueness of the solution, $u(x, y)$, as the following example shows. Let $f(x, y, u, p, q) = -|u|^{\frac{1}{2}}$, $u_{xy} = -|u|^{\frac{1}{2}}$, $u(0, 0) = 0$, $\sigma(x) = 0$, $\tau(x) = 0$. Then $u(x, y) = 0$ and $u(x, y) = \frac{1}{144}(y - x)^4$ are both solutions. However, it is known that if $f(x, y, z, p, q)$ satisfies a Lipschitz condition in z, in addition to satisfying the other hypotheses of Theorem 1, then the solution, $u(x, y)$, is unique. This is proved in [5] for the linear case, but the proof is the same in the non-linear case.

2. *The Approximate Solution*

If f is a function of x, y alone, and is independent of z, p, q, then a solution of CAUCHY's problem is given by

$$u(x, y) = -\int_x^y d\eta \int_x^\eta f(\xi, \eta)\, d\xi + \int_0^x \sigma(\xi)\, d\xi + \int_0^y \tau(\xi)\, d\xi. \tag{2.1}$$

That (2.1) is a solution is verified as follows: Clearly $u(0, 0) = 0$. Also

$$u_x(x, y) = \int_x^y f(x, \eta)\, d\eta + \sigma(x),$$

and so $u_x(x, x) = \sigma(x)$. Similarly

$$u_y(x, y) = -\int_x^y f(\xi, y)\, d\xi + \tau(y);$$

hence $u_y(x, x) = \tau(x)$. Finally

$$u_{xy}(x, y) = u_{yx}(x, y) = f(x, y). \tag{2.2}$$

For a given positive integer n, let the square $\mathfrak{R}$ $(0 \leqq x,\ y \leqq A)$ be subdivided into n^2 rectangles $\mathfrak{R}^n_{j,k}$ $(j=0, \ldots, n-1;\ k=0, \ldots, n-1)$ as follows: Choose $n+1$ real numbers t^n_j $(j=0, \ldots, n)$ such that $0=t^n_0<t^n_1<\cdots<t^n_{n-1}<t^n_n=A$. Then the vertices of $\mathfrak{R}^n_{j,k}$ will be (t^n_j, t^n_k), (t^n_{j+1}, t^n_k), (t^n_j, t^n_{k+1}), and (t^n_{j+1}, t^n_{k+1}).

We shall use the following notation:

$$\mathfrak{R}^n_{j,k} = \{(x,y) : t^n_j < x < t^n_{j+1};\ t^n_k < y < t^n_{k+1}\},$$
$$\underline{\mathfrak{R}^n_{j,k}|} = \{(x,y) : t^n_j < x \leqq t^n_{j+1};\ t^n_k \leqq y < t^n_{k+1}\},$$
$$\overline{\mathfrak{R}}^n_{j,k} = \{(x,y) : t^n_j \leqq x \leqq t^n_{j+1};\ t^n_k \leqq y \leqq t^n_{k+1}\},$$

etc. We shall also let (x_j, y_k) stand for a fixed but arbitrary point in $\mathfrak{R}^n_{j,k}$.

It can be readily verified that formula (2.1) would still give "solutions" to the Cauchy problem for (x, y) in the interior of $\mathfrak{R}^n_{j,k}$ if $f(x, y)$ is allowed to be piece-wise constant on $\mathfrak{R}$ (*i.e.*, to assume a suitable constant value in each square $\mathfrak{R}^n_{j,k}$, and thus to allow the possibility of discontinuities along the lines $x=t_j$ and $y=t_k$); while σ and τ are replaced by piece-wise constants, *i.e.*, constant for $t_j \leqq \xi \leqq t_{j+1}$, $j=0, \ldots, n-1$. This fact will now be used to derive a scheme for the approximation to the solutions of the Cauchy problem which will be of basic importance in proving Theorem 1.

Our "approximate solution" is then

$$u_n(x,y) = -\int_x^y d\eta \int_x^\eta F_n(\xi,\eta)\, d\xi + \int_0^x \Sigma_n(\xi)\, d\xi + \int_0^y T_n(\xi)\, d\xi, \tag{2.3}$$

where the F_n, Σ_n, and T_n are defined by the following expressions (where we have dropped the subscript and superscript n, as we shall continue to do whenever no confusion will result):

$$\sigma_j = \sigma(t_j); \qquad \tau_j = \tau(t_j).$$

$$\Sigma(x) = \begin{cases} \sigma_{k+1} & \text{if } t_k < x \leqq t_{k+1}; \\ \sigma_0 & \text{if } x = t_0. \end{cases}$$

$$T(x) = \begin{cases} \tau_k & \text{if } t_k \leqq x < t_{k+1}; \\ \tau_n & \text{if } x = t_n. \end{cases}$$

$$u_{j,k} = \begin{cases} u_n(t_j, t_k) & \text{if } j \neq k; \\ \int_0^{t_j} \Sigma(\xi)\, d\xi + \int_0^{t_j} T(\xi)\, d\xi & \text{if } j = k. \end{cases}$$

$$v(x,y) = \begin{cases} u_{j+1,k} & \text{if } k > j \text{ and } (x,y) \in \underline{\mathfrak{R}_{j,k}|}; \\ u_{j,j} & \text{if } (x,y) \in \mathfrak{R}_{j,j}; \\ u_{j,k+1} & \text{if } k < j \text{ and } (x,y) \in |\overline{\mathfrak{R}_{j,k}}. \end{cases}$$

$$\lambda(x,y) = \begin{cases} t_{j+1} & \text{if } k > j \text{ and } (x,y) \in \underline{\mathfrak{R}_{j,k}|}; \\ \tfrac{1}{2}(t_j + t_{j+1}) & \text{if } (x,y) \in \mathfrak{R}_{j,j}; \\ t_j & \text{if } k < j \text{ and } (x,y) \in |\overline{\mathfrak{R}_{j,k}}. \end{cases}$$

$$\mu(x,y) = \begin{cases} t_k & \text{if } k > j \text{ and } (x,y) \in \underline{\mathfrak{R}_{j,k}|}; \\ \tfrac{1}{2}(t_j + t_{j+1}) & \text{if } (x,y) \in \mathfrak{R}_{j,j}; \\ t_{k+1} & \text{if } k < j \text{ and } (x,y) \in |\overline{\mathfrak{R}_{j,k}}. \end{cases}$$

$$\pi(x,y)=\begin{cases}\dfrac{u_{j+1,k}-u_{j,k}}{t_{j+1}-t_j} & \text{if } k>j+1 \text{ and } (x,y)\in\underline{\mathfrak{R}_{j,k}}\rvert;\\ \dfrac{u_{j+1,k+1}-u_{j,k+1}}{t_{j+1}-t_j} & \text{if } k<j-1 \text{ and } (x,y)\in\lvert\overline{\mathfrak{R}_{j,k}};\\ 0 & \text{if } (x,y)\in\mathfrak{R}_{j,j};\\ \sigma_j & \text{if } (x,y)\in\underline{\mathfrak{R}_{j-1,j}}\rvert\cup\lvert\overline{\mathfrak{R}_{j,j-1}},\end{cases}$$

$$\varrho(x,y)=\begin{cases}\dfrac{u_{j+1,k+1}-u_{j+1,k}}{t_{k+1}-t_k} & \text{if } k>j+1 \text{ and } (x,y)\in\underline{\mathfrak{R}_{j,k}}\rvert;\\ \dfrac{u_{j,k+1}-u_{j,k}}{t_{k+1}-t_k} & \text{if } k<j-1 \text{ and } (x,y)\in\lvert\overline{\mathfrak{R}_{j,k}};\\ 0 & \text{if } (x,y)\in\mathfrak{R}_{j,j};\\ \tau_j & \text{if } (x,y)\in\underline{\mathfrak{R}_{j-1,j}}\rvert\cup\lvert\overline{\mathfrak{R}_{j,j-1}}.\end{cases}$$

$$F(x,y)=\begin{cases}0 \qquad \text{if } (x,y)\in\mathfrak{R}_{j,j};\\ f[\lambda(x,y),\mu(x,y),\nu(x,y),\pi(x,y),\varrho(x,y)]\end{cases}$$

otherwise.

At first glance our definitions appear to be circular, since $u_n(x,y)$ is defined in terms of F which is itself a function of the $u_{j,k}$'s, which in turn were defined in terms of u_n. However, it is seen that starting from the line $x=y$ the "approximate solution" (2.3) can be continued into successive $\mathfrak{R}_{j,k}$'s in a well defined manner, *i.e.*, such that only previously computed expressions occur as arguments of F, and hence is well defined in all of $\mathfrak{R}$. To show this explicitly consider the following:

If $(x,y)\in\mathfrak{R}_{j,j}$, then by (2.3) $u(x,y)=\int_0^x \Sigma(\xi)\,d\xi+\int_0^y T(\xi)\,d\xi$.

If $j<k$ and $(x,y)\in\underline{\mathfrak{R}_{j,k}}\rvert$, then

$$\begin{aligned}u(x,y)&=-\int_x^y d\eta\int_x^\eta F(\xi,\eta)\,d\xi+\int_0^x\Sigma(\xi)\,d\xi+\int_0^y T(\xi)\,d\xi\\ &=-\int_{t_k}^y d\eta\int_x^{t_{j+1}}F(\xi,\eta)\,d\xi-\\ &\quad-\int_x^{t_k}d\eta\int_x^\eta F(\xi,\eta)\,d\xi+\int_0^x\Sigma(\xi)\,d\xi+\int_0^{t_k}T(\xi)\,d\xi-\\ &\quad-\int_{t_{j+1}}^y d\eta\int_{t_{j+1}}^\eta F(\xi,\eta)\,d\xi+\int_0^{t_{j+1}}\Sigma(\xi)\,d\xi+\int_0^y T(\xi)\,d\xi+\\ &\quad+\int_{t_{j+1}}^{t_k}d\eta\int_{t_{j+1}}^\eta F(\xi,\eta)\,d\xi-\int_0^{t_{j+1}}\Sigma(\xi)\,d\xi-\int_0^{t_k}T(\xi)\,d\xi\\ &=f_{j+1,k}(x-t_{j+1})(y-t_k)+u(x,t_k)+u(t_{j+1},y)-u_{j+1,k},\end{aligned}\tag{2.4}$$

and hence $u(x,y)$ is determined in $\mathfrak{R}_{j,k}$ in terms of functions already computed in adjoining rectangles.

3. *Some Properties of u_n*

From (2.3) it is seen that if $(x, y) \in \mathfrak{R}_{j,k}$

$$\frac{\partial u_n}{\partial x}(x, y) = \int_x^y F(x, \eta)\, d\eta + \Sigma(x),$$

(3.1)

$$\frac{\partial u_n}{\partial y}(x, y) = -\int_x^y F(\xi, y)\, d\xi + T(y).$$

By the definition of F, Σ, T, it is seen that if $(x, y) \in \mathfrak{R}_{j,k}$

$$\frac{\partial^+ u_n}{\partial x}(t_j, y) = \frac{\partial u_n}{\partial x}(x, y) = \frac{\partial^- u_n}{\partial x}(t_{j+1}, y),$$

(3.2)

$$\frac{\partial^+ u_n}{\partial y}(x, t_k) = \frac{\partial u_n}{\partial y}(x, y) = \frac{\partial^- u_n}{\partial y}(x, t_{k+1}),$$

where, as usual,

$$\frac{\partial^+ u_n}{\partial x}(x, y) = \lim_{h \to 0^+} \frac{u_n(x+h, y) - u_n(x, y)}{h},$$

and

$$\frac{\partial^- u_n}{\partial x}(x, y) = \lim_{h \to 0^+} \frac{u_n(x-h, y) - u_n(x, y)}{-h}.$$

Another useful relation is

$$\frac{u_{j+1,k} - u_{j,k}}{t_{j+1} - t_j} = \frac{-1}{t_{j+1} - t_j} \left\{ \int_{t_{j+1}}^{t_k} d\eta \int_{t_{j+1}}^{\eta} d\xi - \int_{t_j}^{t_k} d\eta \int_{t_j}^{\eta} d\xi \right\} F(\xi, \eta) +$$

$$+ \frac{1}{t_{j+1} - t_j} \left\{ \int_0^{t_{j+1}} d\xi - \int_0^{t_j} d\xi \right\} \Sigma(\xi)$$

(3.3)

$$= \frac{1}{t_{j+1} - t_j} \int_{t_{j+1}}^{t_k} d\eta \int_{t_j}^{t_{j+1}} F(\xi, \eta)\, d\xi + \sigma_{j+1}$$

$$= \int_{t_{j+1}}^{t_k} F(x_j, \eta)\, d\xi + \sigma_{j+1}$$

$$= \frac{\partial^- u_n}{\partial x}(t_{j+1}, t_k) = \frac{\partial u_n}{\partial x}(x_j, t_k) = \frac{\partial^+ u_n}{\partial x}(t_j, t_k).$$

We let $\alpha_n = \max_{0 \leq j \leq n-1} |t_j^n - t_{j+1}^n|$. We are going to consider a sequence of subdivisions of $\mathfrak{R}$ such that $\alpha_n \to 0$ as $n \to \infty$, and we wish to show that the sequence $\{u_n\}$ contains a subsequence which converges to a solution of CAUCHY'S problem. As a first step we have the following two theorems.

Theorem 2. *The elements of the sequence $\{u_n\}$ are uniformly bounded.*

Proof. Since $|f(x, y, z, p, q)| \leq M$, the same inequality holds for $F(x, y)$. Similarly since $\sigma(x)$, $\tau(x)$ are continuous on $0 \leq x \leq A$, they are also bounded in absolute value there, and hence so are $\Sigma(x)$ and $T(x)$; we can without loss of generality assume these bounds to be M. Thus for $0 \leqq x,\ y \leqq A$ we have

$$|F(x,y)| \leq M, \quad |\Sigma(x)| \leq M, \quad |T(x)| \leq M.$$

Hence by (2.3) $|u_n(x,y)| \leqq (A^2+2A)M$, q.e.d.

Theorem 3. *The sequence $\{u_n\}$ is an equicontinuous family of functions in the region $0 \leqq x,\ y \leqq A$.*

Proof. We shall consider the case where $x < \bar{x},\ y < \bar{y}$. The treatment of the other three cases (namely $x < \bar{x},\ y > \bar{y}$; $x > \bar{x},\ y < \bar{y}$; and $x > \bar{x},\ y > \bar{y}$) will be clear once this case is disposed of.

$$\begin{aligned} |u_n(\bar{x},\bar{y}) - u_n(x,y)| &\leqq \left\{ \int_{\bar{x}}^{\bar{y}} d\eta \int_{\bar{x}}^{\eta} d\xi - \int_{x}^{y} d\eta \int_{x}^{\eta} d\xi \right\} |F(\xi,\eta)| + \int_{x}^{\bar{x}} |\Sigma(\xi)|\, d\xi + \int_{y}^{\bar{y}} |T(\xi)|\, d\xi \\ &\leqq \left\{ \int_{x}^{\min(\bar{x},y)} d\xi \int_{\xi}^{y} d\eta + \int_{\max(y,\bar{x})}^{\bar{y}} d\eta \int_{\bar{y}}^{\eta} d\xi \right\} |F(\xi,\eta)| + \int_{x}^{\bar{x}} |\Sigma(\xi)|\, d\xi + \int_{y}^{\bar{y}} |T(\xi)|\, d\xi \\ &\leqq M\{A|x-\bar{x}| + A|y-\bar{y}| + |x-\bar{x}| + |y-\bar{y}|\}, \qquad \text{q.e.d.} \end{aligned}$$

Now Theorems 2 and 3 together with ASCOLI's theorem (see Appendix 2) imply that there is a subsequence $\{u_{r_n}\}$ of $\{u_n\}$ which converges uniformly to a continuous function u in the region $\mathfrak{R}$.

In the special case that f is a function of x, y, z only, but not of p or q, it would be a straightforward task to show that u satisfies an integral equation of the form (2.1) but with $f(\xi,\eta)$ replaced by $f[\xi,\eta,u(\xi,\eta)]$. However, instead of carrying out the details of this proof, we shall proceed to the general case where f is a function of all five variables x, y, z, p, q.

The added complication in this more general case is that while the two first partial derivatives of u_n exist in the interior of each region $\mathfrak{R}_{j,k}$ $(0 \leqq j,\ k \leqq n-1)$, the partial with respect to x of u_n need not exist on any of the lines $x = t_k$, and similarly the partial with respect to y of u_n need not exist on the lines $y = t_j$. Since these lines become dense in the region $\mathfrak{R}$ as $n \to \infty$, ASCOLI's theorem gives us no information about the existence of convergent subsequences of $\{\partial u_n/\partial x\}$ and $\{\partial u_n/\partial y\}$.

Before proceeding further we note that by (3.1)

$$\left| \frac{\partial u_n}{\partial x}(x,y) \right| \leqq M\{|y-x|+1\} \leqq M\{A+1\},$$

and similarly for $|\partial u_n/\partial y|$. Hence $\pi(x,y)$ and $\varrho(x,y)$ are uniformly bounded for $(x,y) \in \mathfrak{R}$.

4. *A Useful Inequality*

The following theorem will be of fundamental importance in circumventing the above mentioned difficulty.

Theorem 4. *If:*

a) t is a non-negative integer;

b) $g_0, g_1, \ldots, g_t$ is a sequence of non-negative real numbers;

c) $z_0, z_1, \ldots, z_t$ is a non-decreasing sequence of real numbers;

d) there are numbers $L>0$ *and* $\varepsilon>0$ *such that* $g_k \leqq \varepsilon + L\sum_{\alpha=r}^{k-1} g_\alpha(z_{\alpha+1}-z_\alpha)$ *for* $k-1=r,\ r+1,\dots,t$; *then*

$$g_k \leqq \left\{\prod_{\alpha=r}^{t}[1+L(z_{\alpha+1}-z_\alpha)]\right\}\{\varepsilon+L\,g_r(z_{r+1}-z_r)\}.$$

Proof. We shall show by induction that

$$\varepsilon + L\sum_{\alpha=r}^{k-1} g_\alpha(z_{\alpha+1}-z_\alpha) \leqq \left\{\prod_{\alpha=r}^{k-1}[1+L(z_{\alpha+1}-z_\alpha)]\right\}\{\varepsilon+L\,g_r(z_{r+1}-z_r)\}$$

for $k-1=r,\ r+1,\dots,t$; this will prove the lemma, since $1+L(z_{\alpha+1}-z_\alpha)\geqq 1$ for $\alpha=r,\ r+1,\dots,t$, and $\varepsilon+L\,g_r(z_{r+1}-z_r)\geqq 0$.

For $k-1=r$ the inequality follows from the hypotheses and the fact that $1+L(z_{r+1}-z_r)\geqq 1$ because

$$\varepsilon+L\,g_r(z_{r+1}-z_r)\leqq\{1+L(z_{r+1}-z_r)\}\{\varepsilon+L\,g_r(z_{r+1}-z_r)\}.$$

Proceeding by induction, suppose the inequality holds for $k\leqq t-1$. By the inductive hypotheses and the hypotheses of the theorem,

$$\begin{aligned} g_k &\leqq \varepsilon + L\sum_{\alpha=r}^{k-1} g_\alpha(z_{\alpha+1}-z_\alpha)\\ &\leqq \left\{\prod_{\alpha=r}^{k-1}[1+L(z_{\alpha+1}-z_\alpha)]\right\}\{\varepsilon+L\,g_r(z_{r+1}-z_r)\}. \end{aligned}$$

But then

$$\varepsilon + L\sum_{\alpha=r}^{k} g_\alpha(z_{\alpha+1}-z_\alpha) = \left\{\varepsilon + L\sum_{\alpha=r}^{k-1} g_\alpha(z_{\alpha+1}-z_\alpha)\right\} + L\,g_k(z_{k+1}-z_k),$$

and by use of *d)* this last term becomes

$$\begin{aligned} &\leqq \left\{\prod_{\alpha=r}^{k-1}[1+L(z_{\alpha+1}-z_\alpha)]\right\}\{\varepsilon+L\,g_r(z_{r+1}-z_r)\}\{1+L(z_{k+1}-z_k)\}\\ &= \left\{\prod_{\alpha=r}^{k}[1+L(z_{\alpha+1}-z_\alpha)]\right\}\{\varepsilon+L\,g_r(z_{r+1}-z_r)\}, \qquad \text{q.e.d.} \end{aligned}$$

5. Definition and Some Properties of p_n and q_n

If $(x,y)\in\overline{\Re}_{j,k}$ let

$$\begin{aligned} p_n(x,y) &= \begin{cases} \dfrac{\partial u_n}{\partial x}(x,y) & \text{if } t_j<x<t_{j+1};\\[2mm] \dfrac{\partial^- u_n}{\partial x}(x,y) & \text{if } x=t_j \text{ and } j<k, \text{ or if } x=A;\\[2mm] \dfrac{\partial^+ u_n}{\partial x}(x,y) & \text{if } x=t_j \text{ and } j>k, \text{ or if } x=0; \end{cases}\\ q_n(x,y) &= \begin{cases} \dfrac{\partial u_n}{\partial y}(x,y) & \text{if } t_k<y<t_{k+1};\\[2mm] \dfrac{\partial^+ u_n}{\partial y}(x,y) & \text{if } y=t_k \text{ and } j<k, \text{ or if } y=0;\\[2mm] \dfrac{\partial^- u_n}{\partial y}(x,y) & \text{if } y=t_k \text{ and } j>k, \text{ or if } y=A. \end{cases} \end{aligned}\tag{5.1}$$

We shall show that p_n and q_n, which are defined throughout $\mathfrak{R}$, satisfy the hypotheses of ARZELÀ's theorem (see Appendix 2). As a first step consider $(x, y) \in \mathfrak{R}_{j,k}$ and $(\bar{x}, \bar{y}) \in \mathfrak{R}_{\bar{j},k}$. Then

$$\left|\frac{\partial u_n}{\partial x}(x, y) - \frac{\partial u_n}{\partial x}(\bar{x}, \bar{y})\right| = \left|\frac{\partial^+ u_n}{\partial x}(t_j, y) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, \bar{y})\right|.$$

Now

$$\left|\frac{\partial^+ u_n}{\partial x}(t_j, y) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, \bar{y})\right| \leqq \max_{t_k \leqq y,\, \bar{y} \leqq t_{k+1}} \left|\frac{\partial^+ u_n}{\partial x}(t_j, y) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, \bar{y})\right|.$$

By the linearity of the partial derivatives, the term on the right of the last inequality is the maximum of the four expressions

$$\text{(a)} \quad \left|\frac{\partial^+ u_n}{\partial x}(t_j, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_{k+1})\right|,$$

$$\text{(b)} \quad \left|\frac{\partial^+ u_n}{\partial x}(t_j, t_k) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_k)\right|,$$

$$\text{(c)} \quad \left|\frac{\partial^+ u_n}{\partial x}(t_j, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_k)\right|,$$

$$\text{(d)} \quad \left|\frac{\partial^+ u_n}{\partial x}(t_j, t_k) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_{k+1})\right|.$$

Consider (c). We have

$$\begin{aligned}&\left|\frac{\partial^+ u_n}{\partial x}(t_j, t_{k+1}) - \frac{\partial^+ u_u}{\partial x}(t_{\bar{j}}, t_k)\right| \\ &\qquad \leqq \left|\frac{\partial^+ u_n}{\partial x}(t_j, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_{k+1})\right| + \left|\frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_k)\right|.\end{aligned}$$

By (3.1) and (3.2), the terms in the last absolute value sign equal

$$\left|\left\{\int_{t_{\bar{j}}}^{t_{k+1}} - \int_{t_{\bar{j}}}^{t_k}\right\} F(t_{\bar{j}}, \eta)\, d\eta + \Sigma(t_{\bar{j}}) - \Sigma(t_{\bar{j}})\right| \leqq \int_{t_k}^{t_{k+1}} |F(t_{\bar{j}}, \eta)|\, d\eta \leqq M \cdot |t_{k+1} - t_k| \leqq M\alpha_n.$$

Hence

$$\left|\frac{\partial^+ u_n}{\partial x}(t_j, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_k)\right| \leqq \left|\frac{\partial^+ u_n}{\partial x}(t_j, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_{k+1})\right| + M\alpha_n.$$

A similar result holds for (d). Thus we need only consider the terms (a) plus $M\alpha_n$, or (b) plus $M\alpha_n$. Furthermore (a) arises from (b) by replacing k by $k+1$. Thus

$$\begin{aligned}\left|\frac{\partial u_n}{\partial x}(x, y) - \frac{\partial u_n}{\partial x}(\bar{x}, \bar{y})\right| &\leqq \max_{0 \leqq k \leqq n-1} \left|\frac{\partial^+ u_n}{\partial x}(t_j, t_k) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_k)\right| + M\alpha_n \\ &= \max_{0 \leqq k \leqq n-1} |\pi(x_j, y_k) - \pi(x_{\bar{j}}, y_k)| + M\alpha_n,\end{aligned} \tag{5.2}$$

if $x \leqq y$, $\bar{x} \leqq \bar{y}$. The last equality does not hold if $x > y$ or $\bar{x} > y$ or both, but the inequality between the left side and the right side of (5.2) holds for any $(x, y) \in \mathfrak{R}_{j,k}$ and $(\bar{x}, \bar{y}) \in \mathfrak{R}_{\bar{j},k}$. This can be shown as above by, for example, using (d) in place of (b) if $x \leqq y$ and $\bar{x} > \bar{y}$.

We now proceed to estimate the last expression in absolute value signs.

$$
\begin{aligned}
|\pi(x_{\bar{j}}, y_k) - \pi(x_j, y_k)| &\leqq |\sigma_{\bar{j}+1} - \sigma_{j+1}| + \left| \int_{x_{\bar{j}}}^{y_k} F(x_{\bar{j}}, \eta)\, d\eta - \int_{x_j}^{y_k} F(x_j, \eta)\, d\eta \right| \\
&\leqq |\sigma_{\bar{j}+1} - \sigma_{j+1}| + \left| \int_{t_{\bar{j}}}^{t_k} |F(x_{\bar{j}}, \eta)\, d\eta - \int_{t_j}^{t_k} F(x_j, \eta)\, d\eta \right| + 2M\alpha_n, \\
&\leqq 2M\alpha_n + 2M\{|y_k - t_k| + |y_k - t_{k+1}| + |t_k - t_{k+1}|\} + |\sigma_{\bar{j}+1} - \sigma_{j+1}| + \\
&\quad + \left| \int_{t_j}^{t_{\bar{j}}} |F(x_j, \eta)|\, d\eta \right| + \left| \int_{t_{\bar{j}}}^{t_k} |F(x_{\bar{j}}, \eta) - F(x_j, \eta)|\, d\eta \right|, \\
&\leqq 2M\alpha_n + |\sigma_{\bar{j}+1} - \sigma_{j+1}| + M\cdot|t_{\bar{j}} - t_j| + \\
&\quad + \left| \int_{t_{\bar{j}}}^{t_k} |f[\lambda(x_{\bar{j}}, \eta), \mu(x_{\bar{j}}, \eta), \nu(x_{\bar{j}}, \eta), \pi(x_{\bar{j}}, \eta), \varrho(x_{\bar{j}}, \eta)] - \right. \\
&\quad - f[\lambda(x_j, \eta), \mu(x_j, \eta), \nu(x_j, \eta), \pi(x_{\bar{j}}, \eta), \varrho(x_{\bar{j}}, \eta)] + \\
&\quad + f[\lambda(x_j, \eta), \mu(x_j, \eta), \nu(x_j, \eta), \pi(x_{\bar{j}}, \eta), \varrho(x_{\bar{j}}, \eta)] - \\
&\quad \left. - f[\lambda(x_j, \eta), \mu(x_j, \eta), \nu(x_j, \eta), \pi(x_j, \eta), \varrho(x_j, \eta)]|\, d\eta \right|, \\
&\leqq \Bigl\{2M\alpha_n + |\sigma_{\bar{j}+1} - \sigma_{j+1}| + M|t_{\bar{j}} - t_j| + \\
&\quad + \left| \int_{t_{\bar{j}}}^{t_k} |f[\lambda(x_{\bar{j}}, \eta), \mu(x_{\bar{j}}, \eta), \nu(x_{\bar{j}}, \eta), \pi(x_{\bar{j}}, \eta), \varrho(x_{\bar{j}}, \eta)] - \right. \\
&\quad \left. - f[\lambda(x_j, \eta), \mu(x_j, \eta), \nu(x_k, \eta)\, \pi(x_{\bar{j}}, \eta), \varrho(x_{\bar{j}}, \eta)]|\, d\eta \right| + \\
&\quad + \left| L \int_{t_{\bar{j}}}^{t_k} |\varrho(x_{\bar{j}}, \eta) - \varrho(x_j, \eta)|\, d\eta \right| \Bigr\} + \left| L \int_{t_{\bar{j}}}^{t_k} |\pi(x_{\bar{j}}, \eta) - \pi(x_j, \eta)|\, d\eta \right|.
\end{aligned}
\tag{5.3}
$$

Let y_β be chosen so that $t_\beta < y_\beta < t_{\beta+1}$; then

$$
\left| \int_{t_{\bar{j}}}^{t_k} |\pi(x_{\bar{j}}, \eta) - \pi(x_j, \eta)|\, d\eta \right| = \sum_{\beta=\bar{j}}^{k-1} (t_{\beta+1} - t_\beta)\, |\pi(x_{\bar{j}}, y_\beta) - \pi(x_j, y_\beta)|.
$$

Let γ_1 be the terms in the brace on the right side of (5.3). Then, by use of Theorem 4 with z replaced by t and g_β by $|\pi(x_{\bar{j}}, y_\beta) - \pi(x_j, y_\beta)|$, (5.3) can be written

$$
\begin{aligned}
&|\pi(x_{\bar{j}}, y_k) - \pi(x_j, y_k)| \\
&\qquad \leqq \gamma_1 + L \sum_{\beta=\bar{j}}^{k-1} (t_{\beta+1} - t_\beta)\, |\pi(x_{\bar{j}}, y_\beta) - \pi(x_j, y_\beta)|, \\
&\qquad \leqq \left\{ \prod_{\beta=\bar{j}}^{n-1} [1 + L(t_{\beta+1} - t_\beta)] \right\} \{\gamma_1 + L(t_{\bar{j}+1} - t_{\bar{j}})\, |\pi(x_{\bar{j}}, y_{\bar{j}}) - \pi(x_j, y_{\bar{j}})|\}.
\end{aligned}
\tag{5.4}
$$

By (3.2), (5.2) we have

$$
\begin{aligned}
|p_n(\bar{x}, \bar{y}) - p_n(x, y)| &= \left| \frac{\partial u_n}{\partial x}(\bar{x}, \bar{y}) - \frac{\partial u_n}{\partial x}(x, y) \right| \\
&\leqq \max_{0 \leqq k \leqq n-1} |\pi(x_{\bar{j}}, y_k) - \pi(x_j, y_k)| + M\alpha_n.
\end{aligned}
\tag{5.5}
$$

We can now prove the following:

Theorem 5. *Given* $\varepsilon > 0$, *there exists* $\delta > 0$, $N \geqq 0$ *such that if* $(x, y) \in \mathfrak{R}_{j,k}$, $(\bar{x}, \bar{y}) \in \mathfrak{R}_{\bar{j},k}$ *and* $|x - \bar{x}| < \delta$, $n \geqq N$, *then* $|p_n(\bar{x}, \bar{y}) - p_n(x, y)| \leqq \varepsilon$.

Proof. Choose $\varepsilon' > 0$. We will first show that γ_1 and $|\pi(x_{\bar j}, y_{\bar j}) - \pi(x_j, y_{\bar j})|$ can be made less than ε' by taking n sufficiently large and $|t_j - t_{\bar j}|$ sufficiently small.

$$\begin{aligned} |\pi(x_{\bar j}, y_{\bar j}) - \pi(x_j, y_{\bar j})| &\leqq |\sigma_{\bar j} - \sigma_j| + \left| \int_{t_{\bar j}}^{t_{\bar j+1}} |F(x_{\bar j}, \eta)|\, d\eta - \int_{t_j}^{t_{\bar j+1}} |F(x_j, \eta)|\, d\eta \right| \\ &\leqq |\sigma_{\bar j} - \sigma_j| + 2M|t_{\bar j+1} - t_{\bar j}| + M|t_{\bar j+1} - t_j| \\ &\leqq |\sigma_{\bar j} - \sigma_j| + 2M\alpha_n + M[\alpha_n + |t_{\bar j} - t_j|], \end{aligned}$$

and clearly we can find $\delta' > 0$, $N' \geqq 0$ such that the right side is $< \varepsilon'$ whenever $n \geqq N'$ and $|t_{\bar j} - t_j| < \delta'$.

We next consider γ_1. We have

$$\begin{aligned} \left| \int_{t_{\bar j}}^{t_k} |\varrho(x_{\bar j}, \eta) - \varrho(x_j, \eta)|\, d\eta \right| &\leqq \left| \int_{t_{\bar j}}^{t_k} \left| \int_{t_{\bar j}}^{\eta} F(\xi, \eta)\, d\xi - \int_{t_j}^{\eta} F(\xi, \eta)\, d\xi \right| d\eta \right| + |\tau_{\bar j+1} - \tau_{j+1}| \\ &\leqq \left| \int_{t_{\bar j}}^{t_k} d\eta \int_{t_j}^{t_{\bar j}} |F(\xi, \eta)|\, d\xi \right| + |\tau_{\bar j+1} - \tau_{j+1}|, \\ &\leqq M|t_k - t_{\bar j}|\,|t_{\bar j} - t_j| + |\tau_{\bar j+1} - \tau_{j+1}|. \end{aligned}$$

Thus we see that we can find $\delta'' > 0$, $N'' \geqq 0$ such that whenever $n \geqq N''$, $|t_{\bar j} - t_j| < \delta''$

(a) $L \left| \int_{t_{\bar j}}^{t_k} |\varrho(t_{\bar j}, \eta) - \varrho(t_j, \eta)|\, d\eta \right| \leqq \frac{1}{5}\varepsilon'$,

(b) $|\sigma_{\bar j+1} - \sigma_{j+1}| \leqq \frac{1}{5}\varepsilon'$,

(c) $M \cdot |t_{\bar j+1} - t_{j+1}| \leqq \frac{1}{5}\varepsilon'$,

(d) $2M\alpha_n \leqq \frac{1}{5}\varepsilon'$.

Consider now

$$\begin{aligned} \text{(e)} \quad & |f[\lambda(x_{\bar j}, \eta), \mu(x_{\bar j}, \eta), \nu(x_{\bar j}, \eta), \pi(x_{\bar j}, \eta), \varrho(x_{\bar j}, \eta)] - \\ & - f[\lambda(x_j, \eta), \mu(x_j, \eta), \nu(x_j, \eta), \pi(x_{\bar j}, \eta), \varrho(x_{\bar j}, \eta)]|. \end{aligned}$$

We have already noted that π and ϱ are uniformly bounded for $(x, y) \in \mathfrak{R}$. Also

$$|\nu(x, y)| \leqq \max_{(\xi, \eta) \in \mathfrak{R}} |u_n(\xi, \eta)| \leqq M\{A^2 + 2A\}.$$

Since $f(x, y, z, p, q)$ is uniformly continuous in all five variables for $(x, y) \in \mathfrak{R}$ and z, p, q bounded, we see that we can also make (e) $\leqq \varepsilon'/4A$ for $|x_{\bar j} - x_j| \leqq \delta''$ and $n \geqq N''$ by suitable choice of $\delta'' > 0$ and $N'' > 0$. Thus $\gamma_1 \leqq \varepsilon'$. Hence by (5.4) and (5.5),

$$\begin{aligned} |p_n(\bar{x}, \bar{y}) - p_n(x, y)| &\leqq \left\{ \prod_{\beta = \bar j+1}^{n} [1 + L(t_{\beta+1} - t_\beta)] \right\} \varepsilon' \{1 + L(t_{\bar j+1} - t_{\bar j})\} + M\alpha_n \\ &\leqq \varepsilon' \{1 + L\alpha_n\} \exp\{LA\} + M\alpha_n \end{aligned} \tag{5.6}$$

whenever $n \geqq \max\{N', N''\}$, $|t_{\bar j} - t_j| \leqq \min\{\delta', \delta''\}$. Since ε' can be chosen arbitrarily small, this proves our theorem.

We are now ready to show that the sequence $\{p_n\}$ satisfies the hypotheses of ARZELÀ's theorem.

Theorem 6. *The sequence $\{p_n\}$ is uniformly bounded.*

Proof. $|p_n(x,y)| \leqq \int\limits_x^y |F(x,y)\,d\eta| + |\Sigma(x)| \leqq M|x-y| + M \leqq M(A+1)$, q.e.d.

Theorem 7. *Given $\varepsilon > 0$, there exists $\delta > 0$, $N \geqq 0$ such that*

$$|p_n(\bar x, \bar y) - p_n(x,y)| < \varepsilon$$

whenever $|\bar x - x| < \delta$, $|\bar y - y| < \delta$, $n \geqq N$.

Proof. Let $(x,y) \in \overline{\mathfrak{R}}_{j,k}$, $(\bar x, \bar y) \in \overline{\mathfrak{R}}_{\bar j, \bar k}$. Then

$$|p_n(\bar x, \bar y) - p_n(x,y)| \leqq |p_n(\bar x, \bar y) - p_n(x, \bar y) + |p_n(x,\bar y) - p_n(x,y)|.$$

Now by Theorem 5, given $\varepsilon' > 0$, there exist $\delta' > 0$ and $N' \geqq 0$ such that if $n \geqq N'$, $|\bar x - x| < \delta'$, then $|p_n(\bar x, \bar y) - p_n(x, \bar y)| < \varepsilon'$. Hence

$$\begin{aligned} |p_n(\bar x, \bar y) - p_n(x,y)| &\leqq \varepsilon' + \Big|\int\limits_x^{\bar y} F(x,\eta)\,d\eta - \int\limits_x^{y} F(x,\eta)\,d\eta\Big| \\ &\leqq \varepsilon' + M|\bar y - y| \leqq \varepsilon' + M\delta, \end{aligned}$$

from which the conclusion of the theorem is immediate.

We can now apply ARZELÀ's theorem and conclude that there is a subsequence $\{p_{r'_n}\}$ of $\{p_n\}$ which converges uniformly to a continuous function p in the region $\mathfrak{R}$.

By symmetry it is clear that everything we have done for p_n goes through for q_n, and hence there is a subsequence $\{q_{r''_n}\}$ of $\{q_n\}$ which converges uniformly to a continuous function q in the region $\mathfrak{R}$.

If we first choose the subsequence $\{q_{r''_n}\}$, and then choose $\{p_{r'_n}\}$ to be a subsequence of $\{p_{r''_n}\}$, an finally choose $\{u_{r_n}\}$ to be a subsequence of $\{u_{r'_n}\}$ then we see that

$$\lim_{n\to\infty} u_{r_n} = u, \qquad \lim_{n\to\infty} p_{r_n} = p, \qquad \lim_{n\to\infty} q_{r_n} = q. \tag{5.7}$$

6. Convergence to a Solution

We wish to show that u, the limit of the sequence $\{u_{r_n}\}$, is a solution of the Cauchy problem.

We have already shown that $\{u_n\}$, $\{p_n\}$, and $\{q_n\}$ are uniformly bounded for $(x,y) \in \mathfrak{R}$, *i.e.*, there exists Z, P, Q, all $\geqq 0$ and independent of n, such that $|u_n(x,y)| \leqq Z$, $|p_n(x,y)| \leqq P$, $|q_n(x,y)| \leqq Q$. Also $f(x,y,z,p,q)$ is uniformly continuous for $(x,y) \in \mathfrak{R}$, $|z| \leqq Z$, $|p| \leqq P$, $|q| \leqq Q$. For all $n \geqq 0$ and for $(\xi,\eta) \in \mathfrak{R}$ we have

$$\begin{aligned} \pi(\xi,\eta) &= p_n[\lambda(\xi,\eta), \mu(\xi,\eta)], \\ \varrho(\xi,\eta) &= q_n[\lambda(\xi,\eta), \mu(\xi,\eta)], \\ \nu(\xi,\eta) &= u_n[\lambda(\xi,\eta), \mu(\xi,\eta)] \end{aligned}$$

(the π, ϱ, ν, λ and μ should also carry a subscript n, but we omit it).

Also, by renumbering the elements of our sub-sequences $\{u_{r_n}\}$, $\{p_{r_n}\}$, $\{q_{r_n}\}$, we can write

$$(6.1) \quad \lim_{n\to\infty} u_n(x,y) = u(x,y), \quad \lim_{n\to\infty} p_n(x,y) = p(x,y), \quad \lim_{n\to\infty} q_n(x,y) = q(x,y),$$

the convergence being uniform for $(x,y)\in\mathfrak{R}$.

Consider now

$$u_n(x,y) - \left\{-\int_x^y d\eta \int_x^\eta f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)]\,d\xi + \int_0^x \sigma(\xi)\,d\xi + \int_0^y \tau(\xi)\,d\xi\right\}$$

$$(6.2) \quad \begin{aligned} &= \int_x^y d\eta \int_x^\eta d\xi\,\{f[\xi,\eta,u(\xi,\eta),p(\xi,\eta,q(\xi,\eta))] - \\ &\quad - f[\lambda(\xi,\eta),\mu(\xi,\eta),u_n(\lambda(\xi,\eta),\mu(\xi,\eta)),p_n(\lambda(\xi,\eta)\,\mu(\xi,\eta)),q_n(\lambda(\xi,\eta),\mu(\xi,\eta))]\} \\ &\quad + \int_0^x \{\Sigma(\xi)-\sigma(\xi)\}\,d\xi + \int_0^y \{T(\xi)-\tau(\xi)\}\,d\xi. \end{aligned}$$

Let us write λ,μ for $\lambda(\xi,\eta),\mu(\xi,\eta)$. Then

$$\begin{aligned} &|f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)] - f[\lambda,\mu,u_n(\lambda,\mu),p_n(\lambda,\mu),q_n(\lambda,\mu)]| \\ &\quad \leqq |f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)] - f[\lambda,\mu,u(\lambda,\mu),p(\lambda,\mu),q(\lambda,\mu)]| + \\ &\quad\quad + |f[\lambda,\mu,u(\lambda,\mu),p(\lambda,\mu),q(\lambda,\mu)] - f[\lambda,\mu,u_n(\lambda,\mu),p_n(\lambda,\mu),q_n(\lambda,\mu)]|. \end{aligned}$$

Now by (6.1) and the uniform continuity of f, given $\varepsilon>0$, there exists $N'\geqq 0$ such that if $n\geqq N'$, then

$$|f[\lambda,\mu,u(\lambda,\mu),p(\lambda,\mu),q(\lambda,\mu)] - f[\lambda,\mu,u_n(\lambda,\mu),p_n(\lambda,\mu),q_n(\lambda,\mu)]| \leqq \tfrac{1}{2}\varepsilon$$

for all $(\lambda(\xi,\eta),\mu(\xi,\eta))\in\mathfrak{R}$.

Moreover from the definition of λ,μ it is seen that

$$\lim_{n\to\infty} \lambda(\xi,\eta) = \xi, \qquad \lim_{n\to\infty} \mu(\xi,\eta) = \eta,$$

the convergence being uniform for $(\xi,\eta)\in\mathfrak{R}$. Also u, p, and q are uniformly continuous in $\mathfrak{R}$. Hence there exists an $N''\geqq N'\geqq 0$ such that if $n\geqq N''$ then

$$|f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)] - f[\lambda,\mu,u(\lambda,\mu),p(\lambda,\mu),q(\lambda,\mu)]| \leqq \tfrac{1}{2}\varepsilon.$$

Again, since Σ and T converge uniformly to σ and τ respectively, there exists $N\geqq N''\geqq N'\geqq 0$ such that if $n\geqq N$, $|\Sigma(\xi)-\sigma(\xi)|\leqq\varepsilon$, $|T(\xi)-\tau(\xi)|\leqq\varepsilon$ for all ψ such that $0\leqq\xi\leqq A$. Hence for $n\geqq N$ we have by (6.2)

$$\begin{aligned} &\Bigg|u_n(x,y) - \Bigg\{-\int_x^y d\eta \int_x^\eta f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)]\,d\xi + \\ &\qquad + \int_0^x \sigma(\xi)\,d\xi + \int_0^y \tau(\xi)\,d\xi\Bigg\}\Bigg| \leqq \varepsilon\{A^2+2A\}, \end{aligned}$$

for any $(x,y)\in\mathfrak{R}$. This shows that

$$(6.3) \quad u(x,y) = -\int_x^y d\eta \int_x^\eta f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)]\,d\xi + \int_0^x \sigma(\xi)\,d\xi + \int_0^y \tau(\xi)\,d\xi.$$

Hence we have

$$\frac{\partial u}{\partial x}(x, y) = \int_x^y f[x, \eta, u(x, \eta), p(x, \eta), q(x, \eta)]\, d\eta + \sigma(x),$$

$$\frac{\partial u}{\partial y}(x, y) = -\int_x^y f[\xi, y, u(\xi, y), p(\xi, y), q(\xi, y)]\, d\xi + \tau(y),$$

$$\frac{\partial^2 u}{\partial x\, \partial y}(x, y) = \frac{\partial^2 u}{\partial y\, \partial x}(y, x) = f[x, y, u(x, y), p(x, y), q(x, y)].$$

All that remains to be done is to show that $\partial u/\partial x = p$, $\partial u/\partial y = q$. To this end consider the following, where $\lambda = \lambda(x, \eta)$ and $\mu = \mu(x, \eta)$:

$$\begin{aligned} p_n(x, y) &- \left\{ \int_x^y f[x, \eta, u(x, \eta), p(x, \eta), q(x, \eta)]\, d\eta + \sigma(x) \right\} \\ &= \int_x^y \{ f[\lambda, \mu, u_n(\lambda, \mu), p_n(\lambda, \mu), q_n(\lambda, \mu)] - \\ &\quad - f[x, \eta, u(x, \eta), p(x, \eta), q(x, \eta)]\}\, d\eta + \Sigma(x) - \sigma(x). \end{aligned} \tag{6.4}$$

In the same way as for (6.2) we can show that given $\varepsilon > 0$, there exists an $N' > 0$ such that whenever $n > N'$, the integrand on the right side of (6.4) is in absolute value less than ε. Also if $t_k < x \leqq t_{k+1}$, we have $|\Sigma(x) - \sigma(x)| = |\sigma_{k+1} - \sigma(x)|$, and since σ is uniformly continuous, this will be less than ε for $n \geqq N''$ where $N'' \geqq 0$ is independent of x. Combining these results, we see that

$$\begin{aligned} p(x, y) &= \int_x^y f[x, \eta, u(x, \eta), p(x, \eta), q(x, \eta),]\, d\eta + \sigma(x) \\ &= \frac{\partial u}{\partial x}(x, y). \end{aligned}$$

Since q can be treated in an analogous manner, we have proved that a subsequence of "approximate solutions", $\{u_n\}$, converges to a solution of CAUCHY's problem.

Section II. The Mixed Boundary Value Problem

7. Statement of the Problem

Section II deals with the mixed boundary value problem, the precise formulation of which is given in the following theorem.

Theorem 8. *If f satisfies conditions a), b), c), d) of Theorem 1, and if φ and ψ are functions with together with their first derivatives are continuous real-valued functions of the real variable ξ for $0 \leqq \xi \leqq A$, and $\varphi(0) = \psi(0)$, then there is at least one u such that for $0 \leqq x$, $y \leqq A$:*

(i) *u is a function of x and y;*

(ii) *u and its partial derivatives u_x, u_y and $u_{xy} = u_{yx}$ are continuous for $0 \leqq x$, $y \leqq A$;*

(iii) $u_{xy}(x, y) = f[x, y, u(x, y), u_x(x, y), u_y(x, y)]$;

(iv) $u(0, y) = \varphi(y)$;

(v) $u(x, x) = \psi(x)$.

The solution $u(x, y)$, which exists by Theorem 8, is not necessarily unique, as is shown by the following example. Let $f(x, y, u, p, q) = -4(7y^2 - 4xy)|u|^{\frac{1}{2}}$, $u_{xy} = -4\{7y^2 - 4xy\}|u|^{\frac{1}{2}}$, $u(x, 0) = 0$, $u(x, x) = 0$. Then $u(x, y) \equiv 0$ and $u(x, y) = (y^2 - xy)^4$ are both solutions. However, as in the case of Theorem 1, if $f(x, y, z, p, q)$ satisfies a Lipschitz condition in z, in addition to the other hypotheses of Theorem 8, then the solution is known to be unique. This is proved in [5] for the linear case, but the proof is the same in the non-linear case.

8. *The Approximate Solution*

If f is a function of x, y alone, and is independent of z, p, q then a solution of the mixed problem is given by

$$u(x, y) = \int_x^y d\eta \int_0^x f(\xi, \eta)\, d\xi + \psi(x) - \varphi(x) + \varphi(y). \tag{8.1}$$

That this is so is verified as follows. Clearly $u(0, y) = \varphi(y)$, and $u(x, x) = \psi(x)$. Also

$$u_x(x, y) = \int_x^y f(x, \eta)\, d\eta - \int_0^x f(\xi, x)\, d\xi + \psi'(x) - \varphi'(x),$$

$$u_y(x, y) = \int_0^x f(\xi, y)\, d\xi + \varphi'(y), \tag{8.2}$$

$$u_{xy}(x, y) = u_{yx}(x, y) = f(x, y), \qquad \text{q.e.d.}$$

As in Section I, let f be replaced by a function which is constant on each $\mathfrak{R}^n_{j,k}$. Also let ψ and φ be replaced by functions which are continuous but with piecewise continuous first derivatives, *i.e.*, the first derivatives may have discontinuities at $\xi = t_j$, $j = 0, 1, \ldots, n-1$. Then, again as in Section I, formula (8.1) would still give "solutions" to the mixed problem if $(x, y) \in \mathfrak{R}^n_{j,k}$.

We now proceed as before, introducing the "approximate solution" as

$$u_n(x, y) = \int_x^y d\eta \int_0^x F_n(\xi, \eta; x, y)\, d\xi + \Psi_n(x) - \Phi_n(x) + \Phi_n(y), \tag{8.3}$$

where F_n, Ψ_n, and Φ_n are defined by:

$$\varphi_j = \varphi(t_j); \qquad \psi_j = \psi(t_j);$$

$$u_{j,k} = \begin{cases} u_n(t_j, t_k) & \text{if } j \neq k,\ k \neq 0; \\ \varphi_j & \text{if } k = 0; \\ \psi_j & \text{if } j = k. \end{cases}$$

$$\nu(x, y) = \begin{cases} u_{j,k} & \text{if } k > j \text{ and } (x, y) \in |\underline{\mathfrak{R}_{j,k}}; \\ u_{j,k+1} & \text{if } k \leqq j \text{ and } (x, y) \in |\overline{\mathfrak{R}_{j,k}}. \end{cases}$$

$$\lambda(x, y) = \begin{cases} t_j & \text{if } k > j \text{ and } (x, y) \in |\underline{\mathfrak{R}_{j,k}}; \\ t_j & \text{if } k \leqq j \text{ and } (x, y) \in |\overline{\mathfrak{R}_{j,k}}. \end{cases}$$

$$\mu(x, y) = \begin{cases} t_k & \text{if } k > j \text{ and } (x, y) \in |\underline{\mathfrak{R}_{j,k}}; \\ t_{k+1} & \text{if } k \leqq j \text{ and } (x, y) \in |\overline{\mathfrak{R}_{j,k}}. \end{cases}$$

$$\Psi(x) = \begin{cases} \psi_{k+1} - \left[\dfrac{t_{k+1}-x}{t_{k+1}-t_k}\right](\psi_{k+1}-\psi_k) & \text{if } t_k \leqq x < t_{k+1}; \\ \psi_n & \text{if } x = t_n. \end{cases}$$

$$\Phi(x) = \begin{cases} \varphi_{k+1} - \left[\dfrac{t_{k+1}-x}{t_{k+1}-t_k}\right](\varphi_{k+1}-\varphi_k) & \text{if } t_k \leqq x < t_{k+1}; \\ \varphi_n & \text{if } x = t_n. \end{cases}$$

$$\pi(x,y) = \begin{cases} \dfrac{u_{j+1,k}-u_{j,k}}{t_{j+1}-t_j} & \text{if } j<k \text{ and } (x,y) \in \underline{|\mathfrak{R}_{j,k}}; \\ \dfrac{u_{j+1,k+1}-u_{j,k+1}}{t_{j+1}-t_j} & \text{if } j \geqq k \text{ and } (x,y) \in \overline{|\mathfrak{R}_{j,k}}. \end{cases}$$

$$\varrho(x,y) = \frac{u_{j,k+1}-u_{j,k}}{t_{k+1}-t_k} \quad \text{if } j<k \text{ and } (x,y) \in \underline{|\mathfrak{R}_{j,k}}, \quad \text{or if } j \geqq k \text{ and } (x,y) \in \overline{|\mathfrak{R}_{j,k}}.$$

$$F(x,y;\xi,\eta) = \begin{cases} 0 \quad \text{if } (x,y) \in \mathfrak{R}_{j,j} \text{ and } (\xi,\eta) \in \mathfrak{R}_{j,k}; \\ f[\lambda(x,y), \mu(x,y), \nu(x,y), \pi(x,y), \varrho(x,y)] \quad \text{otherwise.} \end{cases}$$

$$F^*(x,y) = f[\lambda(x,y), \mu(x,y), \nu(x,y), \pi(x,y), \varrho(x,y)].$$

We next derive the analogue of equation (2.4). If $j<k$ and $(x,y) \in \underline{|\mathfrak{R}_{k,k}}$ then by (8.3)

$$\begin{aligned} u(x,y) &= \int_x^y d\eta \int_0^x F\, d\xi + \Psi(x) - \Phi(x) + \Phi(y) \\ &= \int_{t_k}^y d\eta \int_{t_j}^x F\, d\xi + \\ &\quad + \int_{t_j}^y d\eta \int_0^{t_j} F\, d\xi + \Psi_j - \Phi_j + \Phi(y) + \\ &\quad + \int_x^{t_k} d\eta \int_0^x F\, d\xi + \Psi(x) - \Phi(x) + \Phi_k - \\ &\quad - \int_{t_j}^{t_k} d\eta \int_0^{t_j} F\, d\xi - \Psi_j + \Phi_j - \Phi_k \\ &= f_{j,k}(x-t_j)(y-t_k) + u(t_j,y) + u(x,t_k) - u_{j,k}. \end{aligned} \tag{8.4}$$

We should get an analogous expression if $(x,y) \in \overline{|\mathfrak{R}_{j,k}}$ and $j \geqq k$.

9. *Some Properties of u_n*

If $(x,y) \in \mathfrak{R}_{j,k}$ then by (8.3),

$$\begin{aligned} \frac{\partial u_n}{\partial x}(x,y) &= \int_x^y F(x,\eta;x,y)\, d\eta - \int_0^x F(\xi,x;x,y)\, d\xi + \Psi'(x) - \Phi'(x), \\ \frac{\partial u_n}{\partial y}(x,y) &= \int_0^x F(\xi,y;x,y)\, d\xi + \Phi'(y). \end{aligned} \tag{9.1}$$

From the definition of F, Φ, Ψ, we have for $(x, y) \in \mathfrak{R}_{j,k}$

$$\frac{\partial^+ u_n}{\partial x}(t_j, y) = \frac{\partial u_n}{\partial x}(x, y) = \frac{\partial^- u_n}{\partial x}(t_{j+1}, y);$$
$$(9.2)\qquad \frac{\partial^+ u_n}{\partial y}(x, t_k) = \frac{\partial u_n}{\partial y}(x, y) = \frac{\partial^- u_n}{\partial y}(x, t_{k+1}).$$

We also have

$$\frac{u_{j+1,k}-u_{j,k}}{t_{j+1}-t_j} = \frac{1}{t_{j+1}-t_j}\{\Psi(t_{j+1}) - \Psi(t_j) - \Phi(t_{j+1}) + \Phi(t_j)\} +$$
$$+ \frac{1}{t_{j+1}-t_j}\left\{\int_{t_{j+1}}^{t_k} d\eta \int_0^{t_{j+1}} d\xi - \int_{t_j}^{t_k} d\eta \int_0^{t_j} d\xi\right\} F^*(\xi, \eta)$$
$$(9.3)\qquad = \frac{1}{t_{j+1}-t_j}\left\{\int_{t_j}^{t_{j+1}} d\eta \int_0^{t_j} d\xi + \int_{t_{j+1}}^{t_k} d\eta \int_{t_j}^{t_{j+1}} d\xi\right\} F^*(\xi, \eta) +$$
$$+ \frac{\psi_{j+1}-\psi_j}{t_{j+1}-t_j} - \frac{\varphi_{j+1}-\varphi_j}{t_{j+1}-t_j} = \frac{\partial u_n}{\partial x}(x_j, t_k),$$

where, as in Section I, $t_j < x_j < t_{j+1}$.

Similarly we find that if $(x, y) \in \mathfrak{R}_{j,k}$

$$\frac{u_{j,k+1}-u_{j,k}}{t_{k+1}-t_k} = \frac{\partial u_n}{\partial y}(t_j, y_k)$$

where $t_k < y < t_{k+1}$.

As in Section I we choose a sequence of subdivisions of $\mathfrak{R}$ such that

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \max_{0 \leqq j \leqq n-1} |t_j - t_{j+1}| = 0.$$

Since φ, φ', ψ, ψ' are continuous in the closed interval $[0, A]$, these functions are bounded in absolute value there. Let M be a common bound of these four functions and of the absolute value of f. Then M will also be a common bound of $|F|$, $|\Psi|$, $|\Psi'|$, $|\Phi|$, $|\Phi'|$.

We now prove the analogues of Theorems 2 and 3 of Section I.

Theorem 9. *The elements of the sequence $\{u_n\}$ are uniformly bounded in $\mathfrak{R}$.*

Proof. By (8.3), $u_n(x, y) \leqq M\{A+3\}$, q.e.d.

Theorem 10. *The sequence $\{u_n\}$ is an equicontinuous family in $\mathfrak{R}$.*

Proof. As in Theorem 3, assume that $x < \bar{x}$, $y < \bar{y}$. Further let us assume that $x < y$, $\bar{x} < \bar{y}$.

$$|u_n(\bar{x}, \bar{y}) - u_n(x, y)| \leqq |\Psi(\bar{x}) - \Psi(x)| + |\Phi(\bar{x}) - \Phi(x)| + |\Psi(\bar{y}) - \Psi(y)| +$$
$$+ \left|\int_{\bar{x}}^{\bar{y}} d\eta \int_0^{\bar{x}} F(\xi, \eta; x, y)\, d\xi - \int_x^y d\eta \int_0^x F(\xi, \eta; x, y)\, d\xi\right|$$
$$\leqq |\Psi(\bar{x}) - \Psi(x)| + |\Phi(\bar{x}) - \Phi(x)| + |\Psi(\bar{y}) - \Psi(y)| +$$
$$+ \left|\left\{\int_{\bar{x}}^{\bar{y}} d\eta \int_0^{\bar{x}} d\xi - \int_x^y d\eta \int_0^x d\xi\right\} F^*(\xi, \eta)\right| + 2\alpha_n M.$$

Consider the difference of the following double integrals:

$$\begin{aligned}
&\left|\left\{\int_{\bar{x}}^{\bar{y}} d\eta \int_{0}^{\bar{x}} d\xi - \int_{x}^{y} d\eta \int_{0}^{x} d\xi\right\} F^*(\xi,\eta)\right| \\
&\qquad \leqq \left|\left\{\int_{\max(\bar{x},y)}^{\bar{y}} d\eta \int_{0}^{\bar{x}} d\xi + \int_{x}^{\min(\bar{x},y)} d\eta \int_{0}^{x} d\xi\right\} F^*(\xi,\eta)\right| + \left|\int_{x}^{y} d\eta \int_{x}^{\bar{x}} |F^*(\xi,\eta)|\, d\xi\right. \\
&\qquad \leqq MA\,|\bar{y}-y| + MA\,|\bar{x}-x| + MA\,|\bar{x}-x| \\
&\qquad \leqq MA\,\{|\bar{y}-y| + 2|\bar{x}-x|\}.
\end{aligned}$$

Now given $\varepsilon > 0$, there exists $\delta > 0$, $N > 0$ such that if $|\bar{x}-x| < \delta$, $|\bar{y}-y| < \delta$, $n > N$, then $|\Psi(\bar{x}) - \Psi(x)| < \frac{1}{9}\varepsilon$, $|\Phi(\bar{x}) - \Phi(x)| < \frac{1}{9}\varepsilon$, $2\alpha_n M < \frac{1}{3}\varepsilon$, $MA\,\{|\bar{y}-y| + 2|\bar{x}-x|\} < \frac{1}{3}\varepsilon$. Hence $|u_n(\bar{x},\bar{y}) - u_n(x,y)| < \varepsilon$, q.e.d.

We see by (9.1) that

$$\left|\frac{\partial u_n}{\partial y}(x,y)\right| \leqq M\{|x-y| + |x|\} + 2M \leqq 2M\{A+1\},$$

and

$$\left|\frac{\partial u_n}{\partial y}(x,y)\right| \leqq M\{A+1\}.$$

Hence $\pi(x,y)$ and $\varrho(x,y)$ are uniformly bounded for $(x,y) \in \mathfrak{R}$.

The remarks following Theorem 3 in Section I hold unchanged for our present problem.

10. Definition and Some Properties of p_n and q_n

If $(x,y) \in \mathfrak{R}_{j,k}$ let

$$(10.1) \qquad p_n(x,y) = \begin{cases} \dfrac{\partial u_n}{\partial x}(x,y) & \text{if } t_j < x < t_{j+1}; \\[2ex] \dfrac{\partial^+ u_n}{\partial x}(x,y) & \text{if } x = t_j \end{cases}$$

$$(10.1') \qquad q_n(x,y) = \begin{cases} \dfrac{\partial u_n}{\partial x}(x,y) & \text{if } t_k < y < t_{k+1}; \\[2ex] \dfrac{\partial^+ u_n}{\partial y}(x,y) & \text{if } y = t_k \text{ and } j < k; \\[2ex] \dfrac{\partial^- u_n}{\partial y}(x,y) & \text{if } y = t_{k+1} \text{ and } j \geqq k. \end{cases}$$

As in Section I, we shall show that p_n and q_n, which are defined throughout $\mathfrak{R}$, satisfy the conditions of ARZELÀ'S theorem. Assume that $(x,y) \in \mathfrak{R}_{j,k}$ and $(\bar{x},\bar{y}) \in \mathfrak{R}_{\bar{j},k}$. We proceed exactly as in Section I, obtaining

$$\begin{aligned}
&\left|\frac{\partial^+ u_n}{\partial x}(t_j, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_k)\right| \\
&\qquad \leqq \left|\frac{\partial^+ u_n}{\partial x}(t_j, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_{k+1})\right| + \left|\frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_{k+1}) - \frac{\partial^+ u_n}{\partial x}(t_{\bar{j}}, t_k)\right|.
\end{aligned}$$

Using (9.1), (9.2) to estimate the last absolute value, we have

$$\left|\left\{\int_{t_{\bar{j}}}^{t_{k+1}} - \int_{t_{\bar{j}}}^{t_k}\right\} F^*(x,\eta)\,d\eta - \left\{\int_0^{t_{\bar{j}}} - \int_0^{t_{\bar{j}}}\right\} F^*(x,\eta)\,d\eta\right| \leqq \int_{t_k}^{t_{k+1}} |F^*(t_{\bar{j}},\eta)|\,d\eta \leqq M\alpha_n .$$

Continuing as in Section I we find

$$(10.2)\quad \left|\frac{\partial u_n}{\partial x}(\bar{x},\bar{y}) - \frac{\partial u_n}{\partial x}(x,y)\right| \leqq \max_{0\leqq k\leqq n-1} |\pi(x_j,y_k) - \pi(x_{\bar{j}},y_k)| + M\alpha_n ,$$

where, as usual, $t_j < x_j < t_{j+1}$; $t_k < y_k < t_{k+1}$.

Proceeding in the same way, we could show that if $(x, y) \in \mathfrak{R}_{j,k}$ and $(\bar{x},\bar{y}) \in \mathfrak{R}_{j,\bar{k}}$ then

$$(10.2')\quad \left|\frac{\partial u_n}{\partial y}(\bar{x},\bar{y}) = \frac{\partial u_n}{\partial y}(x,y)\right| \leqq \max_{0\leqq k\leqq n-1} |\varrho(x_j,y_k) - \varrho(x_j,y_{\bar{k}})| + M\alpha_n .$$

We next estimate the right side of (10.2′).

$$\begin{aligned}
&|\varrho(x_j,y_{\bar{k}}) - \varrho(x_j,y_k)| \\
&\qquad \leqq |\Phi'(y_{\bar{k}}) - \Phi'(y_n)| + \left|\int_0^{\lambda_j} F(\xi,y_{\bar{k}};x,y_{\bar{k}})\,d\xi - \int_0^{x_j} F(\xi,y_k;x_j,y_k)\,d\xi\right| \\
&\qquad \leqq |\Phi'(y_{\bar{k}}) - \Phi'(y_k)| + 2M\alpha_n + \int_0^{t_j} |F^*(\xi,y_{\bar{k}}) - F^*(\xi,y_k)|\,d\xi \\
(10.3')\quad &\qquad \leqq \Big\{|\Phi'(y_{\bar{k}}) - \Phi'(y_k)| + 2M\alpha_n + \\
&\qquad\quad + \int_0^{t_j} |f[\lambda(\xi,y_{\bar{k}}),\mu(\xi,y_{\bar{k}}),\nu(\xi,y_{\bar{k}}),\pi(\xi,y_{\bar{k}}),\varrho(\xi,y_{\bar{k}})] - \\
&\qquad\quad - f[\lambda(\xi,y_k),\mu(\xi,y_k),\nu(\xi,y_k),\pi(\xi,y_{\bar{k}}),\varrho(\xi,y_{\bar{k}})]|\,d\xi + \\
&\qquad\quad + L\int_0^{t_j} |\pi(\xi,y_{\bar{k}}) - \pi(\xi,y_k)|\,d\xi\Big\} + L\int_0^{t_j} |\varrho(\xi,y_{\bar{k}}) - \varrho(\xi,y_k)|\,d\xi .
\end{aligned}$$

As in Section I, we let γ_1 be the term in braces, and we get

$$\begin{aligned}
&|\varrho(x_j,y_{\bar{k}}) - \varrho(x_j,y_k)| \\
(10.4')\quad &\qquad \leqq \gamma_1 + \sum_{\alpha=0}^{j-1} \{L(t_{\alpha+1} - t_\alpha)\,|\varrho(t_\alpha,y_{\bar{k}}) - \varrho(t_\alpha,y_k)|\} \\
&\qquad \leqq \left\{\prod_{\alpha=0}^{n-1} [1 + L(t_{\alpha+1} - t_\alpha)]\right\}\{\gamma_1 + L(t_1 - t_0)\,|\varrho(t_0,y_{\bar{k}}) - \varrho(t_0,y_k)|\}.
\end{aligned}$$

Theorem 11. *Given* $\varepsilon > 0$, *there exists* $\delta > 0$, $N \geqq 0$ *such that if* $(x, y) \in \overline{\mathfrak{R}}_{j,k}$, $(\bar{x},\bar{y}) \in \overline{\mathfrak{R}}_{j,\bar{k}}$ *and*

$$|\bar{y} - y| < \delta, \qquad n \geqq N, \qquad \text{then} \qquad |q_n(\bar{x},\bar{y}) - q_n(x,y)| \leqq \varepsilon .$$

Proof. We have

$$\begin{aligned}
|q_n(\bar{x},\bar{y}) - q_n(x,y)| &\leqq \max_{0\leqq j\leqq n-1} |\varrho(x_j,y_{\bar{k}}) - \varrho(x_j,y_k)| + M\alpha_n \\
&\leqq \left\{\prod_{\alpha=0}^{n-1} [1 + L(t_{\alpha+1} - t_\alpha)]\right\}\{\gamma_1 + L(t_1 - t_0)\,|\varrho(t_0,y_{\bar{k}}) - \varrho(t_0,y_k)|\}.
\end{aligned}$$

As in the proof of Theorem 5, we show that given $\varepsilon' > 0$, we can make γ_1 and $|\varrho(t_0, y_{\bar k}) - \varrho(t_0, y_k)|$ less than ε' by taking n sufficiently large and $|t_{\bar k} - t_k|$ sufficiently small. In fact

$$|\varrho(t_0, y_{\bar k}) - \varrho(t_0, y_k)| = |\Phi'(y_{\bar k}) - \Phi'(y_k)|$$
$$= \left| \frac{\varphi_{\bar k+1} - \varphi_{\bar k}}{t_{\bar k+1} - t_{\bar k}} - \frac{\varphi_{k+1} - \varphi_k}{t_{k+1} - t_k} \right|.$$

and by the mean value theorem this is

$$\left| \frac{d\varphi}{dy}(t_{\bar k} + \bar\vartheta) - \frac{d\varphi}{dy}(t_k + \vartheta) \right|$$

where $0 < \bar\vartheta < (t_{\bar k+1} - t_{\bar k})$, $0 < \vartheta < (t_{k+1} - t_k)$. But φ' is uniformly continuous in $[0, A]$; hence we can take n large enough so that

$$\left| \frac{d\varphi}{dy}(t_{\bar k} + \bar\vartheta) - \frac{d\varphi}{dy}(t_{\bar k}) \right| < \frac{\varepsilon'}{4},$$

$$\left| \frac{d\varphi}{dy}(t_k + \vartheta) - \frac{d\varphi}{dy}(t_k) \right| < \frac{\varepsilon'}{4}.$$

Hence

$$|\varrho(t_0, y_{\bar k}) - \varrho(t_0, y_k)| \leqq \left| \frac{d\varphi}{dy}(t_{\bar k}) - \frac{d\varphi}{dy}(t_k) \right| + \tfrac{1}{2}\varepsilon',$$

and we can take $|t_{\bar k} - t_k|$ small enough so that the right side is $\leqq \varepsilon'$.

Next consider γ_1. We treat $|\Phi'(y_{\bar k}) - \Phi'(y_k)|$ as above. Also,

$$\int_0^{t_j} |\pi(\xi, y_{\bar k}) - \pi(\xi, y_k)|\, d\xi \leqq \int_0^{t_j} \left| \int_\xi^{y_{\bar k}} F(\xi, \eta; \xi, y_{\bar k})\, d\eta - \int_\xi^{y_k} F(\xi, \eta; \xi, y_k)\, d\eta \right| d\xi$$
$$\leqq 2M\alpha_n |t_j| + \int_0^{t_j} \left| \int_{y_k}^{y_{\bar k}} |F^*(\xi, \eta)|\, d\eta \right| d\xi$$
$$\leqq 2M\alpha_n A + M|y_{\bar k} - y_k| \cdot |t_j|$$
$$\leqq MA\{2\alpha_n + |y_{\bar k} - y_k|\}.$$

We now proceed as in the proof of Theorem 5, getting $\gamma_1 < \varepsilon'$ for suitable choice of n and $|t_{\bar k} - t_k|$.

Hence

$$|q_n(\bar x, \bar y) - q_n(x, y)| \leqq \max_{0 \leqq j \leqq n-1} |\varrho(x_j, y_{\bar k}) - \varrho(x_j, y_k)| + M\alpha_n$$
$$\leqq M\alpha_n + \left\{ \prod_{\alpha=0}^{n-1} [1 + L(t_{\alpha+1} - t_\alpha)] \right\} \varepsilon' \{1 + L(t_1 - t_0)\}$$
$$\leqq M\alpha_n + \varepsilon' \{1 + L\alpha_n\} \exp\{LA\}$$

for n sufficiently large and $|t_{\bar k} - t_k|$ sufficiently small, and this proves the theorem.

We next estimate the right side of (10.2).

$$
\begin{aligned}
&|\pi(x_j, y_k) - \pi(x_{\bar{j}}, y_k)| \\
&\quad \leqq |\Psi'(x_j) - \Psi'(x_{\bar{j}})| + |\Phi'(x_j) - \Phi'(x_{\bar{j}})| \\
&\qquad + \left| \int_{x_j}^{y_k} F(x_j, \eta;\, x_j, y_k)\, d\eta - \int_{x_{\bar{j}}}^{y_k} F(x_{\bar{j}}, \eta;\, x_{\bar{j}}, y_k)\, d\eta \right| + \\
&\qquad + \left| \int_0^{x_j} F(\xi, x_j;\, x_j, y_k)\, d\xi - \int_0^{x_{\bar{j}}} F(\xi, x_{\bar{j}};\, x_{\bar{j}}, y_k)\, d\xi \right| \\
&\quad \leqq |\Psi'(x_j) - \Psi'(x_{\bar{j}})| + |\Phi'(x_j) - \Phi'(x_{\bar{j}})| + \\
&\qquad + \left| \int_{t_j}^{t_k} F^*(x_j, \eta)\, d\eta + \int_{t_{\bar{j}}}^{t_k} F^*(x_{\bar{j}}, \eta)\, d\eta \right| + \\
&\qquad + \int_{t_j}^{t_{j+1}} |F^*(x_j, \eta)|\, d\eta + \int_{t_{\bar{j}}}^{t_{\bar{j}+1}} |F|(x_{\bar{j}}, \eta)|\, d\eta + \\
&\qquad + \left| \int_{t_{k+1}}^{t_k} \{|F^*(x_j, \eta)| + |F^*(x_{\bar{j}}, \eta)|\}\, d\eta \right| + \\
&\qquad + \left| \int_{t_k}^{y_k} \{|F^*(x_j, \eta) + |F^*(x_{\bar{j}}, \eta)|\}\, d\eta \right| + \left| \int_{t_{k+1}}^{y_k} \{|F^*(x_j, \eta) + |F^*(x_{\bar{j}}, \eta)|\}\, d\eta \right| + \\
&\qquad + \left| \int_0^{t_j} F^*(\xi;\, t_j)\, d\xi - \int_0^{t_{\bar{j}}} F^*(\xi, t_{\bar{j}})\, d\xi \right| + \left| \int_{t_j}^{x_j} F^*(\xi, t_j)\, d\xi \right| + \left| \int_{t_{\bar{j}}}^{x_{\bar{j}}} F^*(\xi, t_{\bar{j}})\, d\xi \right| \\
&\quad \leqq |\Psi'(x_{\bar{j}}) - \Psi'(x_j)| + |\Phi'(x_{\bar{j}}) - \Phi'(x_j)| + \\
\text{(10.3)}\quad &\qquad + M\{|t_{j+1} - t_j| + |t_{\bar{j}+1} - t_{\bar{j}}| + 2|t_k - t_{k+1}| + \\
&\qquad + 2|y_k - t_k| + 2|t_{k+1} - y_k| + |x_j - t_j| + |x_{\bar{j}} - t_{\bar{j}}|\} + \\
&\qquad + \left| \int_{t_j}^{t_{\bar{j}}} F^*(x_{\bar{j}}, \eta)\, d\eta \right| + \left| \int_{t_j}^{t_{\bar{j}}} F^*(x_j, \eta)\, d\eta \right| + \\
&\qquad + \int_0^{t_j} |F^*(\xi, t_{\bar{j}}) - F^*(\xi, t_j)|\, d\xi + \left| \int_{t_{\bar{j}}}^{t_k} |F^*(x_{\bar{j}}, \eta) - F^*(x_j, \eta)|\, d\eta \right| \\
&\quad \leqq \Big\{ |\Psi'(x_{\bar{j}}) - \Psi'(x_j)| + |\Phi'(x_{\bar{j}}) - \Phi'(x_j)| + M[10\alpha_n + 2|t_{\bar{j}} - t_j|] + \\
&\qquad + \int_0^{t_j} |F^*(\xi, t_{\bar{j}}) - F^*(\xi, t_j)|\, d\xi \Big\} + \left| \int_{t_{\bar{j}}}^{t_k} |F^*(x_{\bar{j}}, \eta) - F^*(x_j, \eta)|\, d\eta \right| \\
&\quad \leqq \Big\{ |\Psi'(x_{\bar{j}}) - \Psi'(x_j)| + |\Phi'(x_{\bar{j}}) - \Phi'(x_j)| + M[10\alpha_n + 2|t_{\bar{j}} - t_j|] + \\
&\qquad + \int_0^{t_j} |F^*(\xi, t_{\bar{j}}) - F^*(\xi, t_j)|\, d\xi + \\
&\qquad + \Big| \int_{t_{\bar{j}}}^{t_k} |f[\lambda(x_{\bar{j}}, \eta), \mu(x_{\bar{j}}, \eta), \nu(x_{\bar{j}}, \eta), \pi(x_{\bar{j}}, \eta), \varrho(x_{\bar{j}}, \eta)] - \\
&\qquad - f[\lambda(x_j, \eta), \mu(x_j, \eta), \nu(x_j, \eta), \pi(x_{\bar{j}}, \eta), \varrho(x_{\bar{j}}, \eta)]|\, d\eta \Big| + \\
&\qquad + \left| L \int_{t_{\bar{j}}}^{t_k} |\varrho(x_{\bar{j}}, \eta) - \varrho(x_j, \eta)|\, d\eta \right| \Big\} + \left| L \int_{t_{\bar{j}}}^{t_k} |\pi(x_{\bar{j}}, \eta) - \pi(x_j, \eta)|\, d\eta \right|.
\end{aligned}
$$

As in Section I, we let γ_1 be the terms in the last brace, and we get

$$
\begin{aligned}
&|\pi(x_{\bar{j}}, y_k) - \pi(x_j, y_k)| \\
&\leqq \gamma_1 + L \sum_{\beta=\bar{j}}^{k-1} (t_{\beta+1} - t_\beta)\, |\pi(x_{\bar{j}}, y_\beta) - \pi(x_j, y_\beta)| \\
&\leqq \left\{ \prod_{\beta=\bar{j}}^{n} [1 + L(t_{\beta+1} - t_\beta)] \right\} \{\gamma_1 + L(t_{\bar{j}+1} - t_{\bar{j}})\, |\pi(x_{\bar{j}}, y_{\bar{j}}) - \pi(x_j, y_{\bar{j}})|\}.
\end{aligned}
\tag{10.4}
$$

Theorem 12. *Given $\varepsilon > 0$, there exist $\delta > 0$ and $N > 0$ such that if $(x, y) \in \overline{\mathfrak{R}}_{j,k}$, $(\bar{x}, \bar{y}) \in \overline{\mathfrak{R}}_{\bar{j},k}$ and $|x - \bar{x}| < \delta$, $n > N$, then $|p_n(\bar{x}, \bar{y}) - p_n(x, y)| \leqq \varepsilon$.*

Proof.

$$
\begin{aligned}
&|p_n(\bar{x}, \bar{y}) - p_n(x, y)| \\
&\quad \leqq \max_{0 \leqq k \leqq n-1} |\pi(x_{\bar{j}}, y_k) - \pi(x_j, y_k)| + M\alpha_n \\
&\quad \leqq \left\{ \prod_{\beta=\bar{j}}^{n} [1 + L(t_{\beta+1} - t_\beta)] \right\} \{\gamma_1 + L(t_{\bar{j}+1} - t_{\bar{j}})\, |\pi(x_{\bar{j}}, y_{\bar{j}}) - \pi(x_j, y_{\bar{j}})|\} + M\alpha_n.
\end{aligned}
$$

We proceed as with Theorems 5 and 11. Choose $\varepsilon' > 0$.

$$
\begin{aligned}
|\pi(x_{\bar{j}}, y_{\bar{j}}) - \pi(x_j, y_{\bar{j}})| &\leqq \left| \int_0^{x_{\bar{j}}} F(\xi, y_{\bar{j}}; x_{\bar{j}}, y_{\bar{j}})\, d\xi - \int_0^{x_j} F(\xi, y_{\bar{j}}; x_j, y_{\bar{j}})\, d\xi \right| \\
&\leqq 2M\alpha_n + \int_{x_j}^{x_{\bar{j}}} |F(\xi, y_{\bar{j}})\, d\xi| \leqq 2M\alpha_n + M|x_{\bar{j}} - x_j|
\end{aligned}
$$

and this is $< \varepsilon'$ for n sufficiently large and $|x_{\bar{j}} - x_j|$ sufficiently small.

Now there exists $\delta' > 0$, $N' > 0$ such that whenever $n > N'$, $|x_{\bar{j}} - x_j| < \delta'$, then

$$
|\Psi'(x_{\bar{j}}) - \Psi'(x_j)| < \tfrac{1}{12}\varepsilon', \qquad |\Phi'(x_{\bar{j}}) - \Phi'(x_j)| < \tfrac{1}{12}\varepsilon',
$$

which is shown in the same way as in the proof of Theorem 11; and

$$
L \left| \int_{t_{\bar{j}}}^{t_k} |\varrho(x_{\bar{j}}, \eta) - \varrho(x_j, \eta)|\, d\eta \right| \leqq \tfrac{1}{12}\varepsilon',
$$

which is shown as in the proof of Theorem 5; and the expression (e) in the proof of Theorem 5 is $\leqq \frac{1}{12A}\varepsilon'$, and similarly if the η and $x_{\bar{j}}$ are interchanged; and

$$
M[10\alpha_n + 2|t_{\bar{j}} - t_j|] < \tfrac{1}{12}\varepsilon'.
$$

Hence

$$
\begin{aligned}
\gamma_1 \leqq \tfrac{1}{2}\varepsilon' + \int_0^{t_j} |F^*(\xi, t_j) - F^*(\xi, t_j)|\, d\xi &\leqq \tfrac{1}{2}\varepsilon' + \tfrac{1}{12}\varepsilon' \\
&+ L \int_0^{t_j} |\pi(\xi, t_{\bar{j}}) - \pi(\xi, t_j)|\, d\xi + L \int_0^{t_j} |\varrho(\xi, t_{\bar{j}}) - \varrho(\xi, t_j)|\, d\xi.
\end{aligned}
$$

As in the proof of Theorem 11, the first integral on the right is

$$
\leqq LMA\{2\alpha_n + |y_{\bar{k}} - y_k|\}.
$$

By Theorem 11, the integrand in the last expression can be made $< \frac{1}{12A}\varepsilon'$ for proper choice of $|t_{\bar{j}} - t_j|$.

Combining all this, we see that $\gamma_1 < \varepsilon'$ whenever $n > N'$, $|t_{\bar{j}} - t_j| < \delta'$. Hence

$$|p_n(\bar{x},\bar{y}) - p_n(x,y)| \leqq \max_{0 \leqq k \leqq n-1} |\pi(x_{\bar{j}}, y_\beta) - \pi(x_j, y_\beta)| + M\alpha_n$$

$$\leqq \left\{ \prod_{\beta=\bar{j}}^{n} [1 + L(t_{\beta+1} - t_\beta)]\, \varepsilon' \right\} \{\gamma_1 + L(t_{\bar{j}+1} - t_{\bar{j}})\} + M\alpha_n$$

$$\leqq M\alpha_n + \varepsilon'\{1 + L\alpha_n\} \exp\{LA\}$$

or n sufficiently large and $|t_{\bar{j}} - t_j|$ sufficiently small, which proves the theorem.

Theorem 13. *The sequences $\{p_n\}$ and $\{q_n\}$ are uniformly bounded.*

Proof. $$|p_n(x,y)| \leqq \left| \int_x^y |F^*(x,\eta)|\, d\eta \right| + \int_0^x |F^*(\xi,x)|\, d\xi + |\Psi'(x) + \Phi'(x)|$$

$$\leqq M\{|x-y| + |x| + 2\} \leqq 2M\{A+1\};$$

and

$$|q_n(x,y)| \leqq \int_0^x |F^*(\xi,y)|\, d\xi + |\Phi'(y)| \leqq M\{A+1\}, \qquad \text{q.e.d.}$$

Theorem 14. *Given $\varepsilon > 0$, there exist $\delta > 0$ and $N > 0$ such that*

$$|p_n(\bar{x},\bar{y}) - p_n(x,y)| < \varepsilon, \qquad |q_n(\bar{x},\bar{y}) - q_n(x,y)| < \varepsilon$$

whenever $|\bar{x} - x| < \delta$, $|\bar{y} - y| < \delta$, $n > N$.

Proof. Let $(x,y) \in \overline{\mathfrak{R}}_{j,k}$, $(x,y) \in \overline{\mathfrak{R}}_{\bar{j},\bar{k}}$. Then

$$|p_n(\bar{x},\bar{y}) - p_n(x,y)| \leqq |p_n(\bar{x},\bar{y}) - p_n(x,\bar{y})| + |p_n(x,\bar{y}) - p_n(x,y)|.$$

By Theorem 12, given $\varepsilon' > 0$, there exist $\delta' > 0$ and $N' > 0$ such that if $n > N'$, $|\bar{x} - x| < \delta'$ then $|p_n(\bar{x},\bar{y}) - p_n(x,\bar{y})| < \varepsilon'$. Hence

$$|p_n(\bar{x},\bar{y}) - p_n(x,y)| \leqq \varepsilon' + \left| \int_x^{\bar{y}} F^*(x,\eta)\, d\eta - \int_x^{y} F^*(x,\eta)\, d\eta \right| + 2M\alpha_n$$

$$\leqq \varepsilon' + M|\bar{y} - y| + 2M\alpha_n,$$

which proves the theorem for $\{p_n\}$.

Similarly

$$|q_n(\bar{x},\bar{y}) - q_n(x,y)| \leqq |q_n(\bar{x},\bar{y}) - q_n(\bar{x},y)| + |q_n(\bar{x},y) - q_n(x,y)|.$$

We apply Theorem 11 to show that the first term on the right is $< \varepsilon'$ for $n > N'$, $|\bar{y} - y| < \delta'$. The second term is

$$\leqq \left| \int_0^{\bar{x}} F^*(\xi,y)\, d\xi - \int_0^{x} F^*(\xi,y)\, d\xi \right| + 2M\alpha_n \leqq M\{|\bar{x} - x| + 2\alpha_n\},$$

which essentially completes the proof.

As in Section I, ARZELÀ's theorem now guarantees the existence of subsequences such that

$$\lim_{n\to\infty} u_{r_n} = u, \qquad \lim_{n\to\infty} p_{r_n} = p, \qquad \lim_{n\to\infty} q_{r_n} = q, \tag{10.5}$$

the convergence being uniform in $\mathfrak{R}$, and the limit functions being uniformly continuous in $\mathfrak{R}$.

11. *Convergence to a Solution*

We proceed as in § 6. However, instead of (6.2) we have

$$u_n(x,y) - \left\{ \int_x^y d\eta \int_0^x f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)]\,d\xi + \psi(x) - \varphi(x) + \varphi(y) \right\}$$

$$(11.1) \qquad = -\int_x^y d\eta \int_0^x d\xi \{f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)] - F_n(\xi,\eta;x,y)\} + \\ + \Psi_n(x) - \psi(x) - \Phi_n(x) + \varphi(x) + \Phi_n(y) - \varphi(y).$$

With the same notation as in Section I,

$$|f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)] - F_n(\xi,\eta;x,y)| \\ \leqq |f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)] - f[\lambda,\mu,u_n(\lambda,\mu),p_n(\lambda,\mu),q_n(\lambda,\mu)]| + M\alpha_n^2,$$

and the term in absolute value signs on the right can be treated as in Section I, giving the result that for any $\varepsilon > 0$, there exists $N' > 0$ such that if $n \geqq N'$, then

$$\left| \int_x^y d\eta \int_0^x d\xi \{f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)] - F_n(\xi,\eta;x,y)\} \right| \leqq \varepsilon A^2 + M\alpha_n^2.$$

Moreover, if $t_k \leqq x < t_{k+1}$, then

$$|\Psi_n(x) - \psi(x)| = \left| \psi_{k+1} - \left(\frac{t_{k+1} - x}{t_{k+1} - t_k}\right)(\psi_{k+1} - \psi_k) - \psi(x) \right|,$$

and by the uniform continuity of ψ, there exists an $N'' > 0$ such that if $n \geqq N''$, then $|\psi(x) - \psi_{k+1}| < \frac{1}{2}\varepsilon$, $|\psi_{k+1} - \psi_k| < \frac{1}{2}\varepsilon$. Hence for $n > N''$ $|\Psi_n(x) - \psi(x)| < \varepsilon$, and similarly for $|\Phi_n(x) - \varphi(x)|$.

Therefore for $n \geqq N = \max\{N', N''\}$

$$\left| u_n(x,y) - \left\{ \int_x^y d\eta \int_0^x f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)]\,d\xi + \psi(x) - \varphi(x) + \varphi(y) \right\} \right| \\ \leqq \varepsilon\{A^2 + 3\} + M\alpha_n^2,$$

for any $(x,y) \in \mathfrak{R}$, which shows that

$$(11.2) \quad u(x,y) = \int_x^y d\eta \int_0^x f[\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta)]\,d\xi + \psi(x) - \varphi(x) + \varphi(y).$$

Therefore

$$\frac{\partial u}{\partial x}(x,y) = \int_x^y f[x,\eta,u(x,\eta),p(x,\eta),q(x,\eta)]\,d\eta - \\ - \int_0^x f[\xi,x,u(\xi,x),p(\xi,x),q(\xi,x)]\,d\xi + \psi'(x) - \varphi'(x);$$

$$\frac{\partial u}{\partial y} = \int_0^x f[\xi,y,u(\xi,y),p(\xi,y),q(\xi,y)]\,d\xi + \varphi'(y)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} = f[x,y,u(x,y),p(x,y),q(x,y)].$$

We still must show that $\partial u/\partial x = p$, $\partial u/\partial y = q$, but the proof of this is essentially the same as in Section I.

Appendix 1. Numerical applications

We apply the "approximate solutions" given by formulae (2.3) and (8.3) to the equation of LIOUVILLE [*4*]:

(a) $$u_{xy} = \exp\{2\lambda u\}, \qquad \lambda = \text{constant}.$$

The general solution of this equation is

(b) $$u = \frac{1}{2\lambda} \ln \left\{ \frac{g'(y)\, h'(x)}{[\lambda g(y) + h(x)]^2} \right\}$$

where g and h are arbitrary functions having suitable differentiability properties, and such that u, u_x, and u_y exist. This can be readily checked since

(c) $$u_x = \frac{1}{2\lambda} \left\{ \frac{h''(x)}{h'(x)} - 2 \frac{h'(x)}{\lambda g(y) + h(x)} \right\},$$

(d) $$u_y = \frac{1}{2\lambda} \left\{ \frac{g''(y)}{g'(y)} - 2\lambda \frac{g'(y)}{\lambda g(y) + h(x)} \right\},$$

$$u_{xy} = u_{yx} = \frac{g'(y)\, h'(x)}{[\lambda g(y) + h(x)]^2} = \exp\{2\lambda u\}.$$

The numerical process used in the proof of Theorem 1 will now be applied to compute numerically the particular solution of equation (a) which is obtained by taking $\lambda = 1$, $h(x) = \exp x$ and $g(y) = \sin y$. The fact that the solution is actually known in this case will serve as a check on the computations. The values of $\sigma(\xi) = u_x(\xi, \xi)$ and of $\tau(\xi) = u_y(\xi, \xi)$ can be computed from (c) and (d) respectively. For our example we let R be the square with vertices at $(0, 0)$, $(0, \pi/4)$, $(\pi/4, 0)$ and $(\pi/4, \pi/4)$, and we subdivide $\mathfrak{R}$ into n^2 squares.

By (2.4)

(e) $$u_{j,j} = \sum_{j=0}^{n-1} \{\Sigma(t_{j+1}) + T(t_j)\}\,(t_{j+1} - t_j)$$

and for $j < k$

(f) $$u_{j,k+1} = (t_j - t_{j+1})\,(t_{k+1} - t_k)\, f_{j+1,k} + u_{j,k} + u_{j+1,k+1} - u_{j+1,k}.$$

Using (e) and (f) to compute $u(0, \frac{1}{4}\pi)$ we get:

for $n = 2$, $u(0, \frac{1}{4}\pi) = -0.7097$;
for $n = 5$, $u(0, \frac{1}{4}\pi) = -0.7092$;
for $n = 10$, $u(0, \frac{1}{4}\pi) = -0.7088$;
for $n = 20$, $u(0, \frac{1}{4}\pi) = -0.7085$.

On the other hand, from (b) we have to 4 decimal places $u(0, \frac{1}{4}\pi) = -0.7082$.

Next we apply the numerical process used in the proof of Theorem 8 to compute numerically the particular solution of equation (a) which is obtained by taking $\lambda = 1$, $h(x) = \exp x$ and $g(y) = \sin y + 2y$. We can then obtain $u(0, y)$ and $u(x, x)$ from equation (b). We let be the square with vertices at $(0, 0)$, $(0, 2\pi)$, $(2\pi, 0)$ and $(2\pi, 2\pi)$, and we subdivide $\mathfrak{R}$ into n^2 squares. By (8.4), if $j < k$

(g) $$u_{j+1,k+1} = (t_{j+1} - t_j)\,(t_{k+1} - t_k)\, f_{j,k} + u_{j,k+1} + u_{j+1,k} - u_{j,k}.$$

Using (g) to compute $u(\pi, 2\pi)$ we get:

$$\begin{aligned} &\text{for } n=2, && u(\pi,2\pi)=-1.533;\\ &\text{for } n=5, && u(\pi,2\pi)=-1.473;\\ &\text{for } n=10, && u(\pi,2\pi)=-1.461;\\ &\text{for } n=20, && u(\pi,2\pi)=-1.458.\end{aligned}$$

On the other hand, from (b) we have to 4 significant figures $u(\pi, 2\pi)=-1.454$.

Appendix 2. Arzelà's theorem

Arzelà's theorem. *Let* $\{f_n\}$ *be a sequence of real-valued functions, defined on a compact subset* $\mathfrak{K}$ *of a separable metric space, such that:*

(a) *The sequence* $\{f_n(x)\}$ *is uniformly bounded in absolute value on* $\mathfrak{K}$.

(b) *Given any* $\varepsilon>0$, *there exists* $N>0$, $\delta>0$ *such that whenever* $n>N$, $|x_1-x_2|<\delta$, $x_1\in\mathfrak{K}$, $x_2\in\mathfrak{K}$ *then* $|f_n(x_1)-f_n(x_2)|<\varepsilon$. *Then there exists a subsequence* $\{f_{r_n}\}$ *of* $\{f_n\}$ *which converges uniformly to a continuous function* g *defined on* $\mathfrak{K}$.

Proof. Since $\mathfrak{K}$ is separable, there exists a countable set $B\subset\mathfrak{K}$ such that the closure of B is $\mathfrak{K}$. Let $N=\{x_1, x_2, \ldots\}$. Choose a subsequence of $\{f_n\}$ which converges on B as follows. Assume we have chosen a subsequence $\{f_{n,j}\}$ of $\{f_n\}$ which converges at $x_1, x_2, \ldots, x_j$. Consider the point x_{j+1}. Since $|f_{n,j}(x_{j+1})|$ is bounded there will be a subsequence $\{f_{n,j+1}\}$ of $\{f_{n,j}\}$ which converges at x_1, $x_2, \ldots, x_j, x_{j+1}$. Consider a new sequence $\{f_{n,n}\}$ which is formed by a diagonal process from the subsequences $\{f_{n,1}\}, \{f_{n,2}\}, \ldots$. This diagonal sequence converges on B, *i.e.*,

$$\lim_{n\to\infty} f_{n,n}(x_j)=g(x_j)$$

for all $x_j\in B$.

We now show that $\{f_{n,n}\}$ converges uniformly on $\mathfrak{K}$. By hypothesis (b) for any $\varepsilon>0$, there exists $\delta>0$ and $N>0$ such that if $x, y\in\mathfrak{K}$, $|x-y|<\delta$ and $n>N$ then

$$|f_{n,n}(x)-f_{n,n}(y)|<\tfrac{1}{3}\varepsilon. \tag{1}$$

Also, since $\mathfrak{K}$ is compact, for any $\delta>0$ there exists a finite set $\{y_1, \ldots, y_p\}\in B$ such that for any $x\in\mathfrak{K}$ we have $|x-y_j|<\delta$ for some j such that $1\leqq j\leqq p$. We can take N large enough so that in addition to (1) we have for $n, m>N$

$$|f_{n,n}(y_j)-f_{m,m}(y_j)|<\tfrac{1}{3}\varepsilon, \qquad j=1,2,\ldots,p.$$

Then for some j $(1\leqq j\leqq p)$ we have

$$\begin{aligned}|f_{n,n}(x)-f_{m,m}(x)| &\leqq f_{n,n}(x)-f_{n,n}(y_j)|+|f_{n,n}(y_j)-f_{m,m}(y_j)|+|f_{m,m}(y_j)-f_{m,m}(x)|\\ &<\varepsilon,\end{aligned}$$

which proves the uniform convergence of $\{f_{n,n}\}$ to a function g defined on $\mathfrak{K}$. The continuity of g follows directly from hypothesis (b).

Ascoli's theorem is obtained from Arzelà's theorem by specializing hypothesis (b) to read: "The functions $\{f_r\}$ are equicontinuous on $\mathfrak{K}$."

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